Hilbert Function Spaces on the Unit Disk

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Introduction

Let X be a non-empty set. A *reproducing kernel Hilbert space* (*RKHS* for short, or alternatively - a *Hilbert function space*) is a vector space of complex valued functions on X, such that:

- H is a Hilbert space.
- The linear functional on H of point evaluation is bounded. This means that for all $x \in X$, the following functional is bounded:

$$\delta_x : H \to \mathbb{C}, \delta_x(f) = f(x)$$

Since the given functional is bounded, then by the Riesz representation theorem, for all $x \in X$, there is a unique $k_x \in H$, such that:

$$\delta_x f = f(x) = \langle f, k_x \rangle.$$

The function k_x is called the *reproducing kernel at the point x*.

Important example of an RKHS

In this project we focused on studying the family of *weighted Hardy spaces* of analytic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$. Given a sequence $w = (w_n)_{n=0}^{\infty}$ of positive numbers, we define:

$$H_w = \left\{ \sum_{n=0}^{\infty} a_n z^n : \mathbb{D} \to \mathbb{D} \middle| \sum_{n=0}^{\infty} |a_n|^2 w_n < \infty \right\}$$

Two particularly important cases are the classical Hardy space, where $w_n = 1$, and the Bergman space, where $w_n = \frac{1}{n+1}$.

Let H be an RKHS on the disk and $A \subset \mathbb{D}$, we define a *finite subspace* to be:

$$H_A = \text{span}\{k_a : a \in A\}$$

Definition - We say that a bijective bounded linear map $T : H_1 \to H_2$ is an *isomorphism*, if there is some bijection $\phi : X_1 \to X_2$ and a non-vanishing function $\lambda(s) : X_1 \to \mathbb{C}$, such that T maps kernel functions into 'scaled' kernel functions:

$$T(k_s) = \lambda(s) k_{\phi(s)}$$

If the isomorphism T is also an isometry, we say that H_1, H_2 are *isometrically isomorphic*.

Two main problems

- *Problem 1* - Can we classify the finite subspaces of the classical Hardy and Bergman space up to isometric isomorphism using geometric properties of A?
- *Problem 2* - Classify weighted Hardy spaces up to isomorphism and up to isometric isomorphism.

Classification of finite subspaces

Theorem 1- If H is the classical Hardy or Bergman space then H_A is isometric to H_B if and only if there exists a conformal automorphism of the disk taking A to B.

The proof proceeds by analyzing properties of the *Pick matrix* of $f : A \to B$ defined as:

$$P(f, A) = \left[(1 - f(x)\overline{f(y)})\langle k_x, k_y \rangle\right]_{(x,y) \in A \times A}$$

Where k_a is the reproducing kernel at $a \in A$. Pick matrices encode many properties of functions. The key step in proving the stated classification theorem is showing that the minors of $P(f, A)$ encode how far f is from being an isometry with respect to pseudo-hyperbolic metric on the disk.

Of course, If we drop the requirement that isometries preserve the RKHS structure, the only invariant of closed subspaces of an RKHS is the dimension, because any two abstract Hilbert spaces of the same dimension are isometric. This changes if we instead look at the multiplier algebras of subspaces.

Definition - Let H be an RKHS. The *multiplier algebra* of H is defined as:

$$M(H) = \{f : X \to \mathbb{C} | fh \in H, \forall h \in H\}$$

Theorem 2 - For the classical Hardy space, we have that $M(H_A)$ and $M(H_B)$ are isometric as abstract Banach algebras if and only if there exists a conformal automorphism of the disk taking A to B.

Classification of Hardy spaces

It turns out that isomorphisms can be better understood through their adjoints, which obtain the particularly simple form of *weighted composition operators*:

$$T^* h(z) = M_f C_\phi h(z) = f(z) h(\phi(z))$$

Where $f(z)$ is a non-vanishing complex valued function on the disk, and $\phi(z)$ is a bijection of the disk. Particularly, we focused on the case where T^* is simply a scalar multiple of a composition operator (i.e, $f = const$). For this case we got:

Theorem 3 - Two weighted Hardy spaces H_w, H_u are isomorphic if and only if the sequences which define them, $w = (w_n)_{n=0}^{\infty}, u = (u_n)_{n=0}^{\infty}$, are 'comparable'. Meaning, there are some $c > 0, M > 0$ such that:

$$0 < c < \frac{w_n}{u_n} < M, \forall n \in \mathbb{N}$$

Theorem 4 - H_w and H_u are isometrically isomorphic if and only if there exists $c > 0$ such that:

$$\frac{w_n}{u_n} = c$$

Conclusions and future ideas

During the week we classified some RKHS's on the disk and their subspaces up to isomorphism or isometry. We found that these isomorphism classes encode faithfully information about sequences of weights and subsets of the disk. This is one of the first steps in exploring the connection between the geometry of a space and families of RKHS's which one can construct on it.

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