Introduction

Let $X$ be a non-empty set. A reproducing kernel Hilbert space (RKHS for short, or alternatively - a Hilbert function space) is a vector space of complex valued functions on $X$, such that:

- $H$ is a Hilbert space.
- The linear functional on $H$ of point evaluation is bounded. This means that for all $x \in X$, the following functional is bounded:
  $$\delta_x : H \to \mathbb{C}, \delta_x(f) = f(x)$$

Since the given functional is bounded, then by the Riesz representation theorem, for all $x \in X$, there is a unique $k_x \in H$, such that:
$$\delta_x f = (x, k_x)$$

The function $k_x$ is called the reproducing kernel at the point $x$.

Important example of an RKHS

In this project we focused on studying the family of weighted Hardy spaces of analytic functions on the unit disk $D \subset \mathbb{C}$. Given a sequence $w = (w_n)_{n=0}^\infty$ of positive numbers, we define:
$$H_w = \left\{ \sum_{n=0}^\infty a_n z^n : D \to \mathbb{D} \mid \sum_{n=0}^\infty |a_n|^2 w_n < \infty \right\}$$

Two particularly important cases are the classical Hardy space, where $w_n = 1$, and the Bergman space, where $w_n = \frac{1}{n+1}$.

Let $B$ be an RKHS on the disk and $A \subset D$, we define a finite subspace to be:
$$H_A = \text{span}\{k_a : a \in A\}$$

Definition - We say that a bijective bounded linear map $T : H_1 \to H_2$ is an isomorphism, if there is some bijection $\phi : X_1 \to X_2$ and a non-vanishing function $\lambda(s) : X_1 \to \mathbb{C}$, such that $T$ maps kernel functions into ‘scaled’ kernel functions:
$$T(k_{x_1}) = \lambda(s) k_{\phi(x_1)}$$

If the isomorphism $T$ is also an isometry, we say that $H_1, H_2$ are isometrically isomorphic.

Two main problems

- **Problem 1** - Can we classify the finite subspaces of the classical Hardy and Bergman space up to isometric isomorphism using geometric properties of $A$?
- **Problem 2** - Classify weighted Hardy spaces up to isomorphism and up to isometric isomorphism.

Classification of finite subspaces

**Theorem 1** - If $H$ is the classical Hardy or Bergman space then $H_A$ is isometric to $H_B$ if and only if there exists a conformal automorphism of the disk taking $A$ to $B$.

The proof proceeds by analyzing properties of the Pick matrix of $A : B \to B$ defined as:
$$P(f, A) = \left\{ (1 - f(x) \overline{f(y)}) (k_x, k_y) \right\}_{x,y \in A \times A}$$

Where $k_a$ is the reproducing kernel at $a \in A$. Pick matrices encode many properties of functions. The key step in proving the stated classification theorem is showing that the minors of $P(f, A)$ encode how far $f$ is from being an isometry with respect to pseudo-hyperbolic metric on the disk.

Of course, if we drop the requirement that isometries preserve the RKHS structure, the only invariant of closed subspaces of an RKHS is the dimension, because any two abstract Hilbert spaces of the same dimension are isometric. This changes if we instead look at the multiplier algebras of subspaces.

**Definition** - Let $H$ be an RKHS. The multiplier algebra of $H$ is defined as:
$$M(H) = \{ f : X \to \mathbb{C} \mid fh \in H, \forall h \in H \}$$

**Theorem 2** - For the classical Hardy space, we have that $M(H_A)$ and $M(H_B)$ are isometric as abstract Banach algebras if and only if there exists a conformal automorphism of the disk taking $A$ to $B$.

Classification of Hardy spaces

It turns out that isomorphisms can be better understood through their adjoints, which obtain the particularly simple form of weighted composition operators:
$$T^*h(z) = M_f C_h(z) = f(z)h(\phi(z))$$

Where $f(z)$ is a non-vanishing complex valued function on the disk, and $\phi(z)$ is a bijection of the disk. Particularly, we focused on the case where $T^*$ is simply a scalar multiple of a composition operator (i.e, $f = \text{const}$). For this case we got:

**Theorem 3** - Two weighted Hardy spaces $H_w, H_u$ are isomorphic if and only if the sequences which define them, $w = (w_n)_{n=0}^\infty$, $u = (u_n)_{n=0}^\infty$, are ‘comparable’. Meaning, there are some $c > 0$, $M > 0$ such that:
$$0 < c < \frac{u_n}{w_n} < M, \forall n \in \mathbb{N}$$

**Theorem 4** - $H_w$ and $H_u$ are isometrically isomorphic if and only if there exists $c > 0$ such that:
$$\frac{u_n}{w_n} = c$$

Conclusions and future ideas

During the week we classified some RKHS’s on the disk and their subspaces up to isomorphism or isometry. We found that these isomorphism classes encode faithfully information about sequences of weights and subsets of the disk. This is one of the first steps in exploring the connection between the geometry of a space and families of RKHS’s which one can construct on it.

Acknowledgements

We would like to thank our mentors for their dedication and patience throughout the week, and the organizers of the research week for the wonderful opportunity.