Recent results in operator dilation theory and applications

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Dilations

$A = (A_1, \ldots, A_d) \in B(\mathcal{H})^d$
$B = (B_1, \ldots, B_d) \in B(\mathcal{K})^d$, where $\mathcal{K} \supset \mathcal{H}$

Definition

A is said to be a **compression** of B if

$$A_i = P_{\mathcal{H}} B_i\big|_{\mathcal{H}} \quad \text{for all} \quad i = 1, \ldots d$$

We then say that B is a **dilation** of A, and we write $A \prec B$.

Equivalently

$$B_i = \begin{pmatrix} A_i & * \\ * & * \end{pmatrix}$$

Classical dilation theorems

Theorem (Halmos, 1950)

If $T \in B(\mathcal{H})$ is a contraction, then

$$U = \begin{pmatrix} T & \sqrt{1 - TT^*} \\ \sqrt{1 - T^*T} & -T^* \end{pmatrix}$$

is a unitary dilation of T.

Theorem (Sz.-Nagy, 1953)

If $T \in B(\mathcal{H})$ is a contraction, then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unitary $U \in B(\mathcal{K})$ such that

$$T^m = P_{\mathcal{H}} U^m \big|_{\mathcal{H}} \text{ for all } m \geq 0$$

And more . . .

A new kind of dilation theorem

Theorem (Helton, Klep, McCullough, Schweighofer, 2019)

Fix n and and a real n-dimensional Hilbert space $\mathcal{H}$.
There exists a constant ϑ_n, a Hilbert space $\mathcal{K}$, an isometry $V : \mathcal{H} \to \mathcal{K}$, and a commuting family $\mathcal{C}$ in the unit ball of $B(\mathcal{K})_{sa}$ such that for every contraction $A \in B(\mathcal{H})_{sa}$, there exists $N \in \mathcal{C}$ such that

$$\frac{1}{\vartheta_n} A = V^* N V$$

They also show find the optimal ϑ_n, and show (!)

$$\vartheta_n \sim \frac{\sqrt{\pi n}}{2}$$

The dimension of matrices is fixed at $n \times n$, but the number of matrices being simultaneously dilated is **not** fixed.

A new kind of dilation theorem (reformulated)

Theorem (Helton, Klep, McCullough, Schweighofer, 2019)

Fix n and and a real n-dimensional Hilbert space $\mathcal{H}$. There exists a constant ϑ_n, a Hilbert space $\mathcal{K}$, such that for every con. $A \in B(\mathcal{H})^d_{sa}$ there is a d-tuple of **commuting normal** *contractions $N \in B(\mathcal{K})^d_{sa}$ such that*

$$\frac{1}{\vartheta_n} A \prec N$$

Questions one is led to ask:

- Why?
- Complex numbers?
- Is there a constant independent of $n = \dim \mathcal{H}$?
- Can we obtain sharper control on the joint spectrum of N?
- If $d < \infty$ and $\dim \mathcal{H} < \infty$, can we do with $\dim \mathcal{K} < \infty$?

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Matrix convex sets

We consider nc sets

$$\mathcal{S} = \bigsqcup_{n=1}^{\infty} \mathcal{S}_n \subset \bigsqcup_{n=1}^{\infty} (M_n(\mathbb{C}))^d_{sa}$$

Definition

The set $\mathcal{S}$ is matrix convex if the following conditions hold:

1. If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, then

$$A \oplus B = (A_1 \oplus B_1, \dots, A_d \oplus B_d) \in \mathcal{S}_{m+n}$$

2. If $A \in \mathcal{S}_n$, and $\phi \in UCP(M_n, M_m)$, then

$$\phi(A) = (\phi(A_1), \dots, \phi(A_d)) \in \mathcal{S}_m$$

Matrix convex sets

Equivalent definition

A nc $\mathcal{S}$ set is said to be ***matrix convex*** if whenever $A^{(j)} \in \mathcal{S}_{n_j}$ and $V_j \in M_{n_j,n}$ for $j = 1, \ldots, k$ such that $\sum_{j=1}^k V_j^* V_j = I_n$, then

$$\sum_{j=1}^{k} V_j^* A^{(j)} V_j = \sum_{j=1}^{k} (V_j^* A_1^{(j)} V_j, \ldots, V_j^* A_d^{(j)} V_j) \in \mathcal{S}_n$$

Every level of a matrix convex set is convex: if $A^{(1)}, \ldots, A^{(k)} \in \mathcal{S}_n$, $\sum_j t_j = 1$, put $V_j = \sqrt{t_j} I_n$, then $\sum_{j=1}^k V_j^* V_j = \sum t_j I_n = I_n$, and

$$\sum_j t_j A^{(j)} = \sum_{j=1}^{k} V_j^* A^{(j)} V_j \in \mathcal{S}_n$$

But more is true: Every level is invariant under unitary conjugation, and different levels are also related.

Example: Free spectrahedra

Example

For every $A \in B(\mathcal{H})^d$, we define its free spectrahedron to be the free set

$$\mathcal{D}_A = \left\{ X \in \bigsqcup_{n=1}^{\infty} (M_n(\mathbb{C}))^d_{sa} : \sum_{j=1}^{d} X_j \otimes A_j \leq I \right\}.$$

Every free spectrahedron is matrix convex set.

The set $\mathcal{D}_A(1) \subset \mathbb{R}^d$ is called a spectraderon.

Example

$[-1,1]^d$ is a spectrahedron $= \mathcal{D}_C(1)$, where C as on the board.
Note: $\mathcal{D}_C = d$-tuples of self-adjoint contractions.

Relaxation of the matrix cube problem

Matrix Cube Problem: Given A, determine whether $[-1,1]^d \subseteq \mathcal{D}_A(1)$?

HKM observed that the inclusion problem for free spectrahedra is equivalent to **UCP interpolation problem**, thus more tractable then that for spectrahedra.

HKMS prove for $A \in M_n^d$:

$$[-1,1]^d \subseteq \mathcal{D}_A(1) \Rightarrow \mathcal{D}_C \subseteq \vartheta_n \mathcal{D}_A$$

Indeed: if $X \in \mathcal{D}_C(n)$, then by their dilation theorem $X \prec \vartheta_n N$, and $\sigma(N) \subseteq [-1,1]^d \subseteq \mathcal{D}_A(1)$. So

$$\sum X_j \otimes A_j \prec \vartheta_n \sum N_j \otimes A_j \leq \vartheta_n I$$

Matricial relaxation of the (NP hard) MCP: Test $\mathcal{D}_C \subseteq \vartheta_n \mathcal{D}_A$. If yes, then $[-1,1]^d \subseteq \vartheta_n \mathcal{D}_A(1)$. If not, then $[-1,1]^d \subsetneq \mathcal{D}_A(1)$.

Example: Matrix ranges

Example

For every $A \in B(\mathcal{H})^d$, we define its matrix range to be the free set

$$\begin{aligned}\mathcal{W}(A) &= \cup_n \{\phi(A) : \phi \in UCP(C^*(A), M_n)\} \\ &= \cup_n \{(\phi(A_1), \ldots, \phi(A_d)) : \phi \in UCP(C^*(A), M_n)\}.\end{aligned}$$

The matrix range of a tuple A is always a closed and bounded matrix convex set.

Every bounded and closed matrix convex set $\mathcal{S}$ arises as $\mathcal{W}(A)$ for some $A \in B(\mathcal{H})^d$ (but maybe $\dim \mathcal{H} = \infty$).

Matrix ranges and UCP interpolation

UCP interpolation problem: On the board $\longleftarrow$

Theorem (DDSS '16 $\leftarrow$ Arveson '72; Dual: HKM '13, AG '15, Zalar 17)

Let $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$. There exists a UCP map $OS(A) \to OS(B)$ sending A_i to B_i if and only if $\mathcal{W}(B) \subseteq \mathcal{W}(A)$.

Corollary (Li and Poon, 2011 (for selfadjoint matrices))

If A and B are normal, then there exists a UCP map $OS(A) \to OS(B)$ sending A_i to B_i if and only if $\sigma(B) \subseteq \operatorname{conv} \sigma(A)$.

Our proof: We showed that for a normal tuple A

$$\mathcal{W}(A) = \mathcal{W}^{\min}(\operatorname{conv}(A))$$

Matrix ranges and UCP interpolation (cont.)

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Corollary

Let $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$. There exists a unital completely isometric map $OS(A) \to OS(B)$ sending A_i to B_i if and only if $\mathcal{W}(B) = \mathcal{W}(A)$.

Definition

A tuple $A \in B(\mathcal{H})^d$ is said to be incompressible if there is no subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\mathcal{W}(P_{\mathcal{H}_0} A\big|_{\mathcal{H}_0}) = \mathcal{W}(A)$.

Theorem (Passer-S. 2019)

Let A, B be two ***incompressible*** *d-tuples of compact operators. Then $\mathcal{W}(A) = \mathcal{W}(B)$ if and only if A and B are unitarily equivalent.*

Minimal and maximal matrix convex sets

Given a matrix convex set $\mathcal{S}$, the "first level" $K = \mathcal{S}_1$ is a convex set.
Conversely, over every convex set $K \subseteq \mathbb{C}^d$ there is a matrix convex set $\mathcal{S}$ such that $\mathcal{S}_1 = K$.
In fact, there are minimal and maximal matrix convex sets $\mathcal{W}^{\text{min}}(K)$ and $\mathcal{W}^{\text{max}}(K)$ over every K.
We define $\theta(K)$ to be the minimal constant c such that

$$\mathcal{W}^{\text{max}}(K) \subseteq c\mathcal{W}^{\text{min}}(K)$$

Theorem (Passer-S-Solel, 2017)

(i) $\theta(\overline{\mathbb{B}}_{p,d}) = d^{1-|1/p-1/2|}$.

(ii) $\theta(K) = 1$ *if and only if* K *is a simplex.*

An example: $K = \overline{\mathbb{D}}$

Let $K = \overline{\mathbb{D}} \subset \mathbb{C}$.

$$\mathcal{W}^{\min}(\overline{\mathbb{D}}) = \{X \in \sqcup_n M_n : \|X\| \leq 1\}$$

Why? Because $\mathcal{W}^{\min}(\overline{\mathbb{D}})$ contains $\overline{\mathbb{D}}$, so contains directs sums = all normals with spectrum in $\overline{\mathbb{D}}$, so contains all compressions of unitaries = all contractions.

$\mathcal{W}^{\max}(\overline{\mathbb{D}})$ is the set of all operators such that $\sum_j V_j^* X^{(j)} V_j \in \overline{\mathbb{D}}$ for sequences $V_j \in M_{n_j,1} \cong \mathbb{C}^{n_j}$ satisfying $\sum V_j^* V_j = 1$.

In particular for all $X \in \mathcal{W}^{\max}(\overline{\mathbb{D}})$ and all $\|v\| = 1$,

$$v^* X v = \langle Xv, v \rangle \in \overline{\mathbb{D}}$$

It turns out that

$$\mathcal{W}^{\max}(\overline{\mathbb{D}}) = \text{ All matrices } X \text{ with } \textbf{numerical range } W(X) \subseteq \overline{\mathbb{D}}$$

Minimal and maximal matrix convex sets

The ground level of a matrix convex set is $K \subseteq \mathbb{R}^d$ (compact and convex).

Characterization of $\mathcal{W}^{\text{min}}$

$\mathcal{W}^{\text{min}}(K) = \{T : \exists$ a commuting normal dilation N of T, $\sigma(N) \subseteq K\}$.

$(x_1, \ldots, x_d) \in K \overset{\oplus}{\Rightarrow}$ tuples of diagonal matrices, $\sigma \subseteq K \xRightarrow{conjugation}$ commuting normal tuples, $\sigma \subseteq K \xRightarrow{compression}$ the current definition.

Characterization of $\mathcal{W}^{\text{max}}$

$\mathcal{W}^{\text{max}}(K) = \{T : \sum a_i T_i \leq c \cdot I$ for every real linear inequality $\sum a_i x_i \leq c$ that is satisfied for every $(x_1, \ldots, x_d) \in K\}$.

Dilations via matrix convex sets

Conclusion: for compact and convex K and L, the inclusion

$$\mathcal{W}^{\text{max}}(K) \subseteq \mathcal{W}^{\text{min}}(L)$$

(perhaps with L a multiple of K) is a very general matrix dilation result: "If a tuple of matrices satisfies the linear inequalities that determine K, it has a commuting normal dilation with joint spectrum in L."

Theorem (DDSS, 2016)

Suppose that $K \subseteq \mathbb{R}^d$ where K has nice symmetry or invariance properties (for example $K = -K$). Then

$$\mathcal{W}^{max}(K) \subset d \cdot \mathcal{W}^{min}(K)$$

Alternatively: for $T \in B(\mathcal{H})^d$, if the numerical range $\mathcal{W}_1(T)$

$$\mathcal{W}_1(T) = \{\phi(T) : \phi \in UCP(B(\mathcal{H}), \mathbb{C})\} \subseteq K$$

then T has a normal dilation N with $\sigma(N) \subseteq d \cdot K$.

Normal dilation with norm constraints

Problem

What is the minimal constant C_d, such that for every d-tuple $A = (A_1, \ldots, A_d)$ of contractions, there exists a d-tuple of **commuting normals** $N = (N_1, \ldots, N_d)$ such that

$$A \prec N$$

and $\|N_i\| \leq C_d$ for $i = 1, \ldots, d$?

It is not hard to show that $C_d \leq d$ (HKMS, DDSS).

Known results

Theorem (Passer-S.-Solel 2018)

For every d-tuple $A = (A_1, \ldots, A_d)$ of selfadjoint contractions, there exists a d-tuple of commuting selfadjoints $N = (N_1, \ldots, N_d)$ with $\|N_i\| \leq \sqrt{d}$ for $i = 1, \ldots, d$, such that N is a dilation of A.
Moreover, $\sqrt{d}$ is the optimal constant for selfadjoints.

Corollary: $\mathcal{W}^{\max}([-1,1]^d) \subseteq \sqrt{d} \cdot \mathcal{W}^{\min}([-1,1]^d) \Leftrightarrow \theta(\mathbb{B}_{\infty,d}) = \sqrt{d}$

Theorem (Passer 2018)

For every d-tuple $A = (A_1, \ldots, A_d)$ of contractions, there exists a d-tuple of commuting normal operators $N = (N_1, \ldots, N_d)$ with $\|N_i\| \leq \sqrt{2d}$ for $i = 1, \ldots, d$, such that N is a dilation of A.

Thus

$$\sqrt{d} \leq C_d \leq \sqrt{2d}$$

A modified problem: q-commuting unitaries

With undergrads M. Ben-Efraim and Y. Yifrach we tried to numerically test the "conjecture" that $C_d = \sqrt{d}$.
We found some (randomly chosen) counter-examples, that showed that $C_2 > 1.03 \times \sqrt{2}$.
Rare, structured $\Longrightarrow$ we were led to consider q-**commuting unitaries**, that is, pairs of unitary matrices u, v that satisfy

$$vu = quv$$

for some $q = e^{i\theta} \in \mathbb{T}$. Numerically: $\theta = \frac{2\pi}{3}$ gives a counter example.

Problem

Let $\theta \in \mathbb{R}$ and $q = e^{i\theta}$. What is the minimal constant c_θ, such that for pair u, v of q-commuting unitaries, there exists a pair U, V of **commuting** unitaries, such that $(u, v) \prec (c_\theta U, c_\theta V)$?

Do q-commuting unitaries exist?

Let $q = e^{i\theta}$. On $\ell^2(\mathbb{Z})$, define

$$U_\theta = \begin{pmatrix} \ddots & & & & & \\ & e^{-i\theta} & & & & \\ & & 1 & & & \\ & & & e^{i\theta} & & \\ & & & & e^{i2\theta} & \\ & & & & & \ddots \end{pmatrix} \;,\; V_\theta = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

That is, for $g = (g(m))_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$,

$$U_\theta g(m) = e^{im\theta} g(m) \;,\; V_\theta g(m) = g(m+1)$$

Then

$$V_\theta U_\theta = e^{i\theta} U_\theta V_\theta$$

Where do q-commuting unitaries arise?

C*-algebras (Rieffel 1978): For $q = e^{i\theta}$, and define A_θ to be the **universal C*-algebra generated by two unitaries** u_θ, v_θ satisfying

$$v_\theta u_\theta = q u_\theta v_\theta$$

The family $\{A_\theta : \theta \in \mathbb{R}\}$ — the **rotation algebras** — is a very important family of C*-algebras. $A_0 = C(\mathbb{T}^2)$. For $\theta \neq 2\pi k$ ($k \in \mathbb{Z}$), A_θ is not commutative, and is referred to as the **noncommutative torus**.

Physics (Weyl, ancient): Q, P position and momentum operators, $[Q, P] = i$. Form two unitary semigroups:

$$U(t) = e^{itQ} \ , \ V(t) = e^{itP}$$

For all $s, t \in \mathbb{R}$

$$U(t)V(s) = e^{-ist}V(s)U(t)$$

A Schrödinger operator from q-commuting unitaries

D. Hofstadter, “Energy levels. . . of Bloch electrons . . . ", Phys. Rev. B. 1976

With all these substitutions, our Schrödinger equation turns into a one-dimensional difference equation:

$$g(m+1)+g(m-1)+2\cos(2\pi m\alpha-\nu)g(m)=\epsilon g(m).$$

Letting $2\pi\alpha=\theta$, we seek the spectrum ;-) of the operator $H:\ell^2\to\ell^2$:

$$Hg(m)=g(m+1)+g(m-1)+2\cos(m\theta-\nu)g(m)$$

for $g=(g(m))_{m\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$. Note that

$$H=V_\theta+V_\theta^*+e^{-i\nu}U_\theta+e^{i\nu}U_\theta^*$$

Fun fact: essentially

$$H\approx h_\theta:=v_\theta+v_\theta^*+u_\theta+u_\theta^*$$

Hofstadter's puzzle

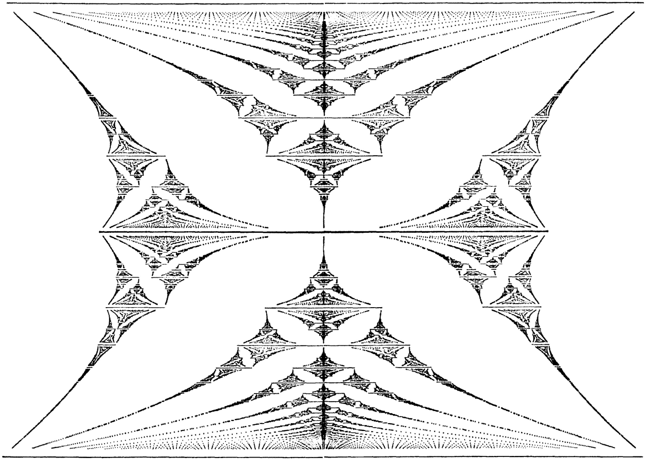
Hofstadter: $\theta = 2\pi\alpha = 2\pi\frac{p}{q} \implies$ spectrum is union of q intervals*.

> ϵ axis (one region centered on each root). This is indeed the case, and is the basis for a very striking (and at first disturbing) fact about this problem: when $\alpha = p/q$, the Bloch band always breaks up into precisely q distinct energy bands. Since small variations in the magnitude of α can produce enormous fluctuations in the value of the denominator q, one is apparently faced with an unacceptable physical prediction. However, nature is ingenious enough to find a way out of this apparent anomaly. Before we go into the resolution,

* - q odd (no relation to "q-commuting", sorry!)

Hofstadter's Butterfly

"...the eye sees something rather continuous." – H.



Spectrum of $h_\theta = u_\theta + u_\theta^* + v_\theta + v_\theta^*$

Problems in the field of rotations algebras

Looking at the butterfly, Hofstadter (and others) made some claims about the spectrum $\sigma(h_\theta)$ of h_θ:

(i) The spectrum is continuous.

(ii) The spectrum is a fractal with some concrete self-similarities.

(iii) For a rational angle $\alpha = \frac{\theta}{2\pi} = \frac{k}{2m+1}$ or $\frac{k}{2m+2}$, the spectrum consists of $2m+1$ intervals.

(iv) For an irrational angle $\alpha = \frac{\theta}{2\pi}$, the spectrum is a Cantor set.

This gave a lot of food for thought for mathematicians for years to come.

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Back to dilation of q-commuting unitaries

Our goal: to dilate q-commuting unitaries to const$\times$commuting unitaries.

Instead: dilate q-commuting unitaries to const$\times$ q'-commuting unitaries.

Theorem (Gerhold and Shalit)

Let $\theta, \theta' \in \mathbb{R}$, set $q = e^{i\theta}, q' = e^{i\theta'}$, and put $c = e^{\frac{1}{4}|\theta - \theta'|}$. Then for any pair of q-commuting unitaries U, V there exists a pair of q'-commuting unitaries U', V' such that $(U, V) \prec (cU', cV')$.

Proof idea:

1) Using Weyl unitaries construct a a particular pair of q'-commuting unitaries. Check that compress to a $c\times q$-commuting unitaries.

2) That's enough! Because Weyl unitaries give the universal representation of the rotation algebras ($+$ Stinespring's theorem).

Proof of the θ to θ' dilation theorem

For a Hilbert space H define the **symmetric Fock space** over H

$$\Gamma(H) := \bigoplus_{k=0}^{\infty} H^{\otimes_s k}$$

Define the **exponential vectors**

$$e(x) := \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} x^{\otimes k}, x \in H$$

they form a linearly independent and total subset of $\Gamma(H)$.
Note: $\langle e(x), e(y) \rangle = e^{\langle x,y \rangle}$ for all $x, y \in H$.
For $z \in H$ we define the **Weyl unitary** $W(z) \in B(\Gamma(H))$ by

$$W(z)e(x) = e(z+x) \exp\left(-\frac{\|z\|^2}{2} - \langle z, x\rangle\right)$$

A simple calculation shows:

$$W(y)W(z) = e^{2i \operatorname{Im}\langle z,y\rangle} W(z)W(y).$$

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Proof: Consider $H \subset K$ with $p : K \to H$, and the symmetric Fock spaces $\Gamma(H) \subset \Gamma(K)$ with $P : \Gamma(K) \to \Gamma(H)$.
Direct calculations show: for $y, z \in K$, the Weyl unitaries $W(y), W(z)$ satisfy:

(1) $W(z), W(y)$ commute up to the phase factor $e^{2i \operatorname{Im}\langle z,y \rangle}$.

(2) $PW(z)\big|_{\Gamma(H)} = e^{-\frac{\|p^\perp z\|^2}{2}} W(pz)$, so it is a scalar multiple of a unitary on $\Gamma(H)$.

(3) $PW(z)\big|_{\Gamma(H)}, PW(y)\big|_{\Gamma(H)}$ commute up to the phase factor $e^{2i \operatorname{Im}\langle pz,py \rangle} = e^{2i \operatorname{Im}\langle z,py \rangle}$.

Remains to show that things can be arranged so that everything works out.

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An application

We wish to study continuity of $\theta \mapsto \sigma(h_\theta)$, where

$$h_\theta = u_\theta + u_\theta^* + v_\theta + v_\theta^*$$

Theorem (AMS*, HR, GS)

Let p be a selfadjoint $*$-polynomial in two noncommuting variables. Then the spectrum $\sigma(p(u_\theta, v_\theta))$ is $\frac{1}{2}$-Hölder continuous in θ with respect to the Hausdorff distance for compact subsets of $\mathbb{C}$.

The issue:

The map $\theta \mapsto h_\theta$ is not continuous in norm:

$$h_\theta - h_{\theta'} = \mathrm{diag}\left((2\cos(m\theta) - 2\cos(m\theta'))_{m\in\mathbb{Z}}\right)$$

thus

$$\limsup_{\theta\to 0} \|h_\theta - h_0\| = 4$$

Our proof (for $p(u_\theta, v_\theta) = h_\theta = u_\theta + u_\theta^* + v_\theta + v_\theta^*$)

$(u_\theta, v_\theta) \prec (cu_{\theta'}, cv_{\theta'})$ where $\theta \approx \theta'$ so $c = e^{\frac{1}{4}|\theta - \theta'|} \approx 1$. Thus:

$$\Rightarrow cu_{\theta'} = \begin{pmatrix} u_\theta & x \\ y & z \end{pmatrix} \quad , \quad \|x\|, \|y\| \leq \sqrt{c^2 - 1} \approx 0$$

Lemma: For selfadjoint operators a, b: $d(\sigma(a), \sigma(b)) \leq \|a - b\|$.
Now $h_\theta = u_\theta + u_\theta^* + v_\theta + v_\theta^*$

$$\Rightarrow ch_{\theta'} = \begin{pmatrix} h_\theta & * \\ * & * \end{pmatrix} \approx \begin{pmatrix} h_\theta & 0 \\ 0 & * \end{pmatrix}$$

$$\sigma(h_\theta) \subseteq \sigma\left(\begin{pmatrix} h_\theta & 0 \\ 0 & * \end{pmatrix}\right) \approx \sigma(ch_{\theta'}) \approx \sigma(h_{\theta'})$$

And conversely.
This generalizes very easily to higher dim. noncommutative tori.

Back to the dilation constants

Problem

Let $\theta \in \mathbb{R}$ and $q = e^{i\theta}$. What is the minimal constant c_θ, such that for pair u, v of q-commuting unitaries, there exists a pair U, V of **commuting** unitaries, such that $(u, v) \prec (c_\theta U, c_\theta V)$?

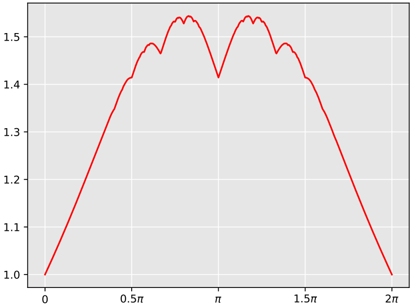
Theorem (Gerhold and Shalit)

For every $\theta \in \mathbb{R}$,

$$c_\theta = \frac{4}{\|u_\theta + u_\theta^* + v_\theta + v_\theta^*\|}$$

That is, for every pair of $e^{i\theta}$-commuting unitaries U, V, there exists a pair U_0, V_0 of commuting unitaries, such that $(u, v) \prec (c_\theta U, c_\theta V)$.

Evaluating the dilation constant



$C_d \geq \max_\theta c_\theta \geq 1.543.$
Conjecture: $\max_\theta c_\theta$ is attained at **silver mean** $\theta_s = \frac{2\pi}{\gamma_s} = 2\pi(\sqrt{2}-1)$, where $\gamma_s = \sqrt{2}+1$ is the **silver ratio**.
Using the rational approximation $\frac{1}{\gamma_s} \approx \frac{2378}{5741}$ with an error of less than $1.1 \cdot 10^{-8}$ and the Lipschitz continuity of c_θ, we found $c_{\theta_s} \approx 1.5437772$.

Thank you for listening!