Dilation theory of CP maps and CP-semigroups

Orr Shalit

Technion

NISER Bhubaneswar, January 2020

References

(i) The notes that I distributed are part of a survey in progress ***Dilation theory: a guided tour***, to appear soon on arxiv and later in Proceeding of IWOTA 2019.

(ii) My blog *Noncommutative Analysis* ->Lecture Notes -> “Topics in Operator Theory 106435". Starts similarly and continues in the operator algebraic direction.

(iii) My forthcoming “paper" with Michael Skeide ***CP-Semigroups and Dilations, Subproduct Systems and Superproduct Systems: The Multiparameter Case and Beyond***.

Completely positive maps

An **operator system** is a selfadjoint unital subspace $1 \in X \subseteq \mathcal{B}(H)$.
A map $\phi : X \to Y$ is said to be **positive** if $X \ni a \geq 0 \implies \phi(a) \geq 0$.
It is **completely positive** if

$$\phi^{(n)} := \phi \otimes \mathrm{id}_{M_n} : M_n(X) = X \otimes M_n \to M_n(Y) = Y \otimes M_n$$

is positive for all $n \in \mathbb{N}$.

Examples of CP maps

- A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between C*-algebras is CP:
$\pi(a^*a) = \pi(a)^*\pi(a)$ so positive, but $\pi^{(n)}$ is also a $*$-homomorphism.
- If $V \in \mathcal{B}(H, K)$ then $\phi : T \mapsto V^*TV$ is completely positive:

$$\phi^{(n)} : (T_{ij}) \mapsto \mathrm{diag}(V^*, \dots, V^*)(T_{ij})\,\mathrm{diag}(V, \dots, V)$$

- Sums of CPs, compositions of CPs, etc.

Stinespring's dilation theorem

Theorem (Stinespring, 1955)

Let $\mathcal{A}$ be a unital C-algebra and let $\phi : \mathcal{A} \to \mathcal{B}(H)$ be a CP map. Then there exists a Hilbert space K, an operator $V \in \mathcal{B}(H, K)$, and a (unital) $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$, such that*

$$\phi(a) = V^* \pi(a) V \quad , \quad \text{for all } a \in \mathcal{A}$$

Moreover, (π, K, V) can be chosen **minimal**, *in the sense that $K = [\pi(\mathcal{A}) V H]$.*

(i) The minimal dilation above is called **the minimal Stinespring dilation**, and it is unique.

(ii) If ϕ is unital then $V^*V = V^*\pi(1)V = \phi(1) = I_H$, so V is an isometry. Then $H \equiv VH$ and write

$$\phi(a) = P_H \pi(a)\big|_H$$

Proof of Stinespring's dilation theorem

Theorem (Stinespring, 1955)

Let $\mathcal{A}$ be a unital C-algebra and let $\phi : \mathcal{A} \to \mathcal{B}(H)$ be a CP map. Then there exists a Hilbert space K, an operator $V \in \mathcal{B}(H, K)$, and a (unital) $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$, such that*

$$\phi(a) = V^*\pi(a)V \quad , \quad \text{for all } a \in \mathcal{A}$$

Proof. Construct the Hilbert space $\mathcal{A} \otimes_\phi H$ obtained from $A \otimes_{alg} H$ with the (semi-)inner product (here is where we use CP)

$$\langle a \otimes g, b \otimes h \rangle = \langle g, \phi(a^*b)h \rangle$$

Define $\pi(a)b \otimes h = ab \otimes h$ and $Vh = 1 \otimes h$ (so $V^*(a \otimes h) = \phi(a)h$).

$$V^*\pi(a)Vh = V^*\pi(a)1 \otimes h = V^*(a \otimes h) = \phi(a)h$$

as required.

Application: representation of CP maps on $\mathcal{B}(H)$

Theorem (Choi-Kraus representation)

Let $\phi : M_n \to M_k$ *be a CP map. Then there exist* $W_i \in M_{n,k}$ *such that*

$$\phi(A) = \sum W_i A W_i^* \quad , \quad A \in M_n$$

(True also for normal CP maps on $\mathcal{B}(H)$).

Proof. $\phi(\cdot) = V^*\pi(\cdot)V$. By basic representation theory of M_n:

$$\pi(A) \cong A \oplus A \oplus \cdots \oplus A = \sum V_i A V_i^*$$

where $V_i : \mathbb{C}^n \to \mathbb{C}^{mn}$ the isometry into the ith summand in $\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n$.
Put $W_i = V^*V_i$.

$$\phi(A) = V^*(\sum V_i A V_i^*)V = \sum W_i A W_i^*$$

Application: a dilation machine

Example: vN inequality $\Rightarrow$ unitary dilation

Suppose that $\|T\| \leq 1$, and that we know

$$\|p(T)\| \leq \|p\|_\infty := \sup_{|z|=1} |p(z)|$$

$C(\mathbb{T}) \supset \mathbb{C}[z] \ni p \mapsto p(T) \in \mathcal{B}(H)$ is unital and contractive.

$$\Longrightarrow C(\mathbb{T}) \supset \mathbb{C}[z] + \overline{\mathbb{C}[z]} \ni p + \overline{q} \mapsto p(T) + q(T)^* \in \mathcal{B}(H)$$

is unital and positive. $\mathbb{C}[z] + \overline{\mathbb{C}[z]}$ is dense in $C(\mathbb{T}) \Rightarrow$ we obtain a unital positive $\phi : C(\mathbb{T}) \to \mathcal{B}(H)$ s.t. $\phi(p) = p(T)$ for polynomials.
$C(\mathbb{T})$ is commutative $\Rightarrow \phi$ is UCP. **Stinespring:** $\pi : C(\mathbb{T}) \to \mathcal{B}(K)$

$$p(T) = \phi(p) = P_H \pi(p)\big|_H = P_H p(\pi(z))\big|_H = P_H p(U)\big|_H$$

$U = \pi(z)$ is unitary because π is a $*$-homomorphism and z is unitary.

A dilation machine

Example: Ando inequality ⇒ Ando dilation?

Suppose that $\|T_1\|, \|T_2\| \leq 1$, and that we know

$$\|p(T_1,T_2)\| \leq \|p\|_\infty := \sup_{|z_1|=|z_2|=1} |p(z_1,z_2)|$$

$C(\mathbb{T}^2) \supset \mathbb{C}[z_1,z_2] \ni p \mapsto p(T_1,T_2) \in \mathcal{B}(H)$ is unital and contractive.

$$\implies \mathbb{C}[z_1,z_2] + \overline{\mathbb{C}[z_1,z_2]} \ni p + \overline{q} \mapsto p(T_1,T_2) + q(T_1,T_2)^* \in \mathcal{B}(H)$$

is unital and positive. $\mathbb{C}[z_1,z_2] + \overline{\mathbb{C}[z_1,z_2]}$ is dense in $C(\mathbb{T}^2)$? No! The argument breaks down. Its true that the map is UCP, but this doesn't help. We need ϕ to be defined on a C*-algebra to use Stinespring's theorem. If we can extend $p \mapsto p(T_1,T_2)$ to a UCP map $\phi : C(\mathbb{T}^2) \to \mathcal{B}(H)$, then we can apply Stinespring as before:

$$p(T_1,T_2) = \phi(p) = P_H\pi(p)\big|_H = P_H p(\pi(z_1),\pi(z_2))\big|_H = P_H p(U_1,U_2)\big|_H$$

$U_i = \pi(z_i)$ is unitary.

Arveson's extension theorem and C*-dilations

Theorem (Arveson, 1969)

Let $X \subset \mathcal{A}$ be an operator system contained in a C-algebra $\mathcal{B}$. Let $\phi : X \to \mathcal{B}(H)$ be a CP map. Then there exists a CP map $\tilde{\phi} : \mathcal{B} \to \mathcal{B}(H)$ such that $\|\tilde{\phi}\| = \|\phi\|$ and which extends ϕ: $\tilde{\phi}(x) = \phi(x)$ for all $x \in X$.*

For a proof, see Paulsen's book (fails for positive). Arveson's theorem can be used together with Stinespring's theorem to otain dilation theorems.

Definition

Let $1 \in X \subseteq \mathcal{B}$ be a unital operator space. A linear map $\phi : X \to \mathcal{B}(H)$ is said to have a **C*-dilation** to $\mathcal{B}$ if there exists a $*$-representation $\pi : \mathcal{B} \to \mathcal{B}(K)$, $K \supseteq H$, such that

$$\phi(x) = P_H \pi(x)\big|_H \quad , \qquad \text{for all } x \in X.$$

Theorem (Arveson, 1969)

Every UCP (or UCC) map has a C-dilation.*

Example: row isometric dilation

A tuple $T = (T_1, \ldots, T_d)$ is said to be a **row contraction** if $\sum T_i T_i^* \leq I$. It is a **row isometry** if $\sum T_i^* T_j = \delta_{ij} I$.

Theorem

Every row contraction has a row isometric dilation.

By this we mean a row isometry (V_i) on $\mathcal{B}(K)$, $K \supset H$, such that

$$T^\alpha = T_{\alpha_1} \cdots T_{\alpha_k} = P_H V_{\alpha_1} \cdots V_{\alpha_k}\big|_H = P_H V^\alpha\big|_H$$

for all $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, 2, \ldots, d\}^k$, for all k.

Example: row isometric dilation

Theorem

Every row contraction has a row isometric dilation.

Proof ($d = 1$). Let $T \in \mathcal{B}(H)$, $\|T\| \leq 1$. Let S be the unilateral shift on ℓ^2. For $r \in (0,1)$, let $D_{rT} = (I - r^2 TT^*)^{1/2}$, and define $K_r(T) : H \to \ell^2 \otimes H$ by

$$K_r(T)h = \sum_n e_n \otimes \left(r^n D_{rT} T^{n*} h\right)$$

$$K_r(T)^* K_r(T)h = \sum r^{2n} T^n D_{rT}^2 T^{n*} h = \sum r^{2n} T^n (I - r^2 TT^*) T^{n*} h =$$

$$= \sum r^{2n} T^n T^{n*} h - \sum r^{2(n+1)} T^{n+1} T^{(n+1)*} h = h$$

$$K_r(T)^* (S \otimes I) K_r(T)h = K_r(T)^* \sum e_{n+1} \otimes \left(r^n D_{rT} T^{n*} h\right) =$$

$$= \sum r^{2n+1} T^{n+1} D_{rT}^2 T^{n*} h = rTh$$

Example: row isometric dilation

Proof continued

Let $T \in \mathcal{B}(H)$, $\|T\| \leq 1$. Let S be the unilateral shift on ℓ^2. On $C^*(S)$ we define a CP map

$$\phi_r(a) = K_r(T)^*(a \otimes I)K_r(T)$$

We saw: $\phi_r(I) = I$, $\phi_r(S) = rT$. Likewise, $\phi_r(S^n) = r^n T^n$. Define a UCP $\Phi := \lim_{r \nearrow 1} \phi_r$

$$\Phi(S^n) = T^n$$

Let $\pi : \mathcal{T} \to \mathcal{B}(K)$ be a C*-dilation of Φ. Then

$$T^n = \Phi(S^n) = P_H \pi(S^n)\big|_H = P_H V^n\big|_H$$

where $V = \pi(S)$ is an isometry, being the image of an isometry under a (unital) $*$-homomorphism.

Dilation theory of completely positive semigroups

The objects of study

$\mathbb{S}$ a semigroup of $\mathbb{R}_+^k$, such that $0 \in \mathbb{S}$.
$T = (T_s)_{s\in\mathbb{S}}$ a family of maps on a unital C*-algebra $\mathcal{B}$.

- T is said to be a **CP-semigroup** (over $\mathbb{S}$) if
  1. T_s is a (contractive) CP map for all s,
  2. $T_0 = \mathbf{id}_{\mathcal{B}}$,
  3. $T_{s+t} = T_s \circ T_t$, for all $s, t \in \mathbb{S}$.
- If $T_s(1) = 1$ for all s, then T is said to be a **Markov semigroup**.
- If T_s is a $*$-endomorphism for all s, then T is said to be an **E-semigroup**.
- Case of greatest interest: $\mathbb{S} = \mathbb{R}_+$, then CP-semigroups $T = (T_t)_{t\geq 0}$ (and E-semigroups) have quantum dynamical interpretations.

$$(\mathbf{UCP}) \quad t \mapsto T_t(a) \qquad \text{evolution in an irreversible quantum system}$$

$$(*\mathbf{auto}) \quad t \mapsto \alpha_t(a) \qquad \text{evolution in a reversible quantum system}$$

The objects of study II

$0 \in \mathbb{S} \subseteq \mathbb{R}_+^k$.
$T = (T_s)_{s\in\mathbb{S}}$ a CP-semigroup on a unital C*-algebra $\mathcal{B}$.

Example

If $T_1, \ldots, T_k$ are k commuting CP maps, then we get a CP-semigroup $(T_s)_{s\in\mathbb{N}^k}$ over $\mathbb{S} = \mathbb{N}^k$:

$$T_s = T_1^{s_1} \circ \cdots \circ T_k^{s_k} \quad \text{where } s = (s_1, \ldots, s_k) \in \mathbb{N}^k.$$

Every CP-semigroup over $\mathbb{S} = \mathbb{N}^k$ arises this way.

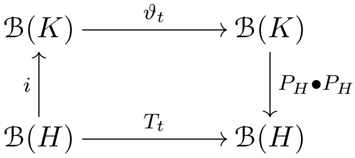
Issue: The Stinespring dilations of different T_s do not work well together.

Bhat's dilation theorem

Theorem (Bhat, 1996)

Let $T = (T_t)_{t \geq 0}$ *be a CP-semigroup on* $\mathcal{B}(H)$*. Then there exists a Hilbert space* K *containing* H*, and an E-semigroup* $\vartheta = (\vartheta_t)_{t \geq 0}$ *on* $\mathcal{B}(K)$*, such that*

$$T_t(A) = P_H \vartheta_t(A) P_H \quad , \quad \textit{for all } t \geq 0 \textit{ and } A \in \mathcal{B}(H).$$



Bhat's dilation theorem

Theorem (Bhat, 1996)

Let $T = (T_t)_{t\geq 0}$ be a CP-semigroup on $\mathcal{B}(H)$. Then there exists a Hilbert space K containing H, and an E-semigroup $\vartheta = (\vartheta_t)_{t\geq 0}$ on $\mathcal{B}(K)$, such that

$$T_t(A) = P_H \vartheta_t(A) P_H \quad , \quad \text{for all } t \geq 0 \text{ and } A \in \mathcal{B}(H).$$

Interpretation

An irreversible quantum dynamical system can be embedded in a reversible one (ϑ can be extended to a group of *-automorphisms).

Application

An index for quantum dynamical semigroups (Bhat).

Other notions of dilations of CP-semigroups have been studied since 70s: Davies, Evans-Lewis, Hudson-Parthasarathy, Kummerer, Sauvageout ...

Bhat's theorem – discrete case (toy version)

Theorem

Let T be a normal CCP map on $\mathcal{B}(H)$. Then there exists a Hilbert space K containing H, and a normal $*$-endomorphism ϑ on $\mathcal{B}(K)$, such that

$$T^n(A) = P_H \vartheta^n(A) P_H \quad , \quad \textit{for all } n \in \mathbb{N} \textit{ and } A \in \mathcal{B}(H).$$

Proof. We know that $T(A) = \sum W_i A W_i^*$. Assume $T(A) = WAW^*$.
$WW^* = T(I) \leq I$ (T is contractive), so W is a contraction.
Let $V \in \mathcal{B}(K)$ be an isometric dilation of W define

$$\vartheta(B) = VBV^* \quad , B \in \mathcal{B}(K)$$

This is an endomorphism:

$$\vartheta(B_1)\vartheta(B_2) = VB_1V^*VB_2V^* = VB_1B_2V^* = \vartheta(B_1B_2)$$

For $A = P_H A P_H \in \mathcal{B}(H)$,

$$P_H \vartheta^n(A) P_H = P_H V^n P_H A P_H V^{n*} P_H = W^n A W^{n*} = T^n(A)$$

Bhat's theorem – discrete case (for real)

Theorem

Let T be a normal CCP map on $\mathcal{B}(H)$. Then there exists a Hilbert space K containing H, and a normal $*$-endomorphism ϑ on $\mathcal{B}(K)$, such that

$$T^n(A) = P_H \vartheta^n(A) P_H \quad , \quad \text{for all } n \in \mathbb{N} \text{ and } A \in \mathcal{B}(H).$$

Proof. We know that $T(A) = \sum W_i A W_i^*$.
$\sum W_i W_i^* = T(I) \leq I$, so $W = (W_i)$ is a row contraction.
Let $V_i \in \mathcal{B}(K)$ be a row isometric dilation of (W_i) define

$$\vartheta(B) = \sum V_i B V_i^* \quad , B \in \mathcal{B}(K)$$

This is an endomorphism (recall $V_i^* V_j = \delta_{ij} I_K$):

$$\vartheta(B_1)\vartheta(B_2) = \sum V_i B_1 V_i^* \sum V_j B_2 V_j^* = \sum V_i B_1 B_2 V_i^* = \vartheta(B_1 B_2)$$

$$P_H \vartheta^n(A) P_H = \sum_{|\alpha|=n} P_H V^\alpha P_H A P_H V^{\alpha *} P_H = \sum_{|\alpha|=n} W^\alpha A W^{\alpha *} = T^n(A)$$

We study the possible generalizations of Bhat's theorem to a CP-semigroup T on a unital C*-algebra $\mathcal{B}$, paramaterized by a semigroup $\mathbb{S} \subseteq \mathbb{R}_+^k$.

Definition

A **dilation** of T is a triple $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A}$ is a C*-algebra, $\vartheta = (\vartheta_s)_{s \in \mathbb{S}}$ is a semigroup of *-endomorphisms, and $p \in \mathcal{A}$ is a projection, such that $\mathcal{B} = p\mathcal{A}p$, and such that

$$T_s(b) = p\vartheta_s(b)p \quad \text{for all } b \in \mathcal{B}, s \in \mathbb{S}.$$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \vartheta_s\ } & \mathcal{A} \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle p \bullet p} \\ \mathcal{B} & \xrightarrow{\ T_s\ } & \mathcal{B} \end{array}$$

Arveson, Bhat, Bhat-Skeide, Markiewicz, Muhly-Solel, Powers, SeLegue, S., S.-Solel, Solel, Vernik,. . .

We study the possible generalizations of Bhat's theorem to a CP-semigroup T on a unital C*-algebra $\mathcal{B}$, paramaterized by a semigroup $\mathbb{S} \subseteq \mathbb{R}_+^k$.

Definition

A **dilation** of T is a triple $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A}$ is a C*-algebra, $\vartheta = (\vartheta_s)_{s \in \mathbb{S}}$ is a semigroup of *-endomorphisms, and $p \in \mathcal{A}$ is a projection, such that $\mathcal{B} = p\mathcal{A}p$, and such that

$$T_s(b) = p\vartheta_s(b)p \quad \text{for all } b \in \mathcal{B}, s \in \mathbb{S}.$$

Questions

1. **Find necessary & sufficient conditions for existence of dilation.**
2. **For fixed k, does every CP-semigroup over $\mathbb{N}^k$ have a dilation?**

.

Key tool: C*-correspondences

Let $\mathcal{B}$ be a C*-algebra. A **Hilbert C*-moudle** over $\mathcal{B}$ is a right module E that has a $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{B}$, such that

(i) $\langle x, x \rangle \geq 0$ for all $x \in E$,

(ii) $\langle x, yb \rangle = \langle x, y \rangle b$ for all $x, y \in E$ and $b \in \mathcal{B}$,

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in E$,

(iv) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$,

(v) $\|x\| := \|\langle x, x \rangle\|^{1/2}$ is a norm on E which makes E into a Banach space.

A **C*-correspondence** is a Hilbert C*-module that also has a left action by adjointable operators.

Tensor product $E \odot F$: obtained from $E \otimes_{alg} F$ by inner product

$$\langle x \otimes y, x' \otimes y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$$

The GNS representation $(\mathcal{E}, \xi)$ of a CP map

Let $T : \mathcal{B} \to \mathcal{B}$ be a CP map. Then there exists a unique C*-correspondence $\mathcal{E}$ over $\mathcal{B}$, and a vector $\xi \in \mathcal{E}$, such that

$$\operatorname{span} \overline{\mathcal{B}\xi\mathcal{B}} = \mathcal{E}$$

and

$$\langle \xi, b\xi \rangle = T(b) \quad \text{for all } b \in \mathcal{B}.$$

Construction: on $\mathcal{E}_0 = \mathcal{B} \otimes_{alg} \mathcal{B}$ put inner product

$$\langle a \otimes b, c \otimes d \rangle = b^* T(a^* c) d$$

and bimodule operation

$$a(x \otimes y)d = ax \otimes yd.$$

Complete the quotient, and put $\xi = 1 \otimes 1$. This works:

$$\langle \xi, b\xi \rangle = \langle 1 \otimes 1, b \otimes 1 \rangle = 1^* T(1^* b) 1 = T(b).$$

The GNS representation $(\mathcal{E}, \xi)$ of a CP map

Let $T : \mathcal{B} \to \mathcal{B}$ be a CP map. Then there exists a unique C*-correspondence $\mathcal{E}$ over $\mathcal{B}$, and a vector $\xi \in \mathcal{E}$, such that

$$\operatorname{span} \overline{\mathcal{B}\xi\mathcal{B}} = \mathcal{E}$$

and

$$\langle \xi, b\xi \rangle = T(b) \quad \text{for all } b \in \mathcal{B}.$$

Construction: on $\mathcal{E}_0 = \mathcal{B} \otimes_{alg} \mathcal{B}$ put inner product

$$\langle a \otimes b, c \otimes d \rangle = b^* T(a^* c) d$$

and bimodule operation

$$a(x \otimes y)d = ax \otimes yd.$$

Complete the quotient, and put $\xi = 1 \otimes 1$. This works:

$$\langle \xi, b\xi \rangle = \langle 1 \otimes 1, b \otimes 1 \rangle = 1^* T(1^* b) 1 = T(b).$$

The GNS representation of a CP-semigroup

Let $T = (T_s)_{s \in \mathbb{S}}$ be a CP-semigroup on $\mathcal{B}$.
For every s, let $(\mathcal{E}_s, \xi_s)$ be the GNS representation of T_s.
For $s, t \in \mathbb{S}$, define

$$w_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \odot \mathcal{E}_t$$

by

$$w_{s,t} : a\xi_{s+t}b \mapsto a\xi_s \odot \xi_t b,$$

and then extend linearly. We check:

$$\langle a\xi_s \odot \xi_t b, a\xi_s \odot \xi_t b \rangle = \langle \xi_t b, \langle a\xi_s, a\xi_s \rangle \xi_t b \rangle = b^* \langle \xi_t, T_s(a^*a)\xi_t \rangle b =$$

$$= b^* T_t(T_s(a^*a))b = b^* T_{t+s}(a^*a)b = \langle a\xi_{s+t}b, a\xi_{s+t}b \rangle.$$

$w_{s,t}$ **is an isometry!**

Subproduct systems[1]

Definition

A **subproduct system** is a family $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences, together with a family $\{w_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \odot \mathcal{E}_t\}$ of isometric bimodule maps, which iterate associatively, i.e., the following diagram is commutative $(\forall r, s, t)$:

$$\begin{array}{ccc} \mathcal{E}_{r+s+t} & \longrightarrow & \mathcal{E}_r \odot \mathcal{E}_{s+t} \\ \big\downarrow & & \big\downarrow \\ \mathcal{E}_{r+s} \odot \mathcal{E}_t & \longrightarrow & \mathcal{E}_r \odot \mathcal{E}_s \odot \mathcal{E}_t \end{array}$$

A **product system** is a subproduct system in which $w_{s,t}$ are all unitaries.

Definition

A family $\{\xi_s \in \mathcal{E}_s\}_{s\in\mathbb{S}}$ is called a **unit** if $w_{s,t}\xi_{s+t} = \xi_s \odot \xi_t$ for all s, t.

[1] **Inclusion systems** by Bhat-Mukherjee; recall the talk by Vijay Kumar U. Introduced also by S.-Solel.

Recap

Subproduct system: $\mathcal{E}_s \odot \mathcal{E}_t \supseteq \mathcal{E}_{s+t}$
Product system: $E_s \odot E_t = E_{s+t}$
Unit: $\xi_s \odot \xi_t = \xi_{s+t}$

For every CP-semigroup on $\mathcal{B}$, there exists a subproduct system $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences (called the **GNS subproduct system**) and a unit $(\xi_s)_{s\in\mathbb{S}}$ such that

$$T_s(b) = \langle \xi_s, b\xi_s \rangle \quad \text{for all } s \in \mathbb{S}, b \in \mathcal{B}.$$

Theorem (S.-Skeide, following Bhat-Skeide, 2000)

Let T be a Markov semigroup. If the GNS subproduct system of T can be embedded in a **product system***, then T has a unital dilation $(\mathcal{A}, \vartheta, p)$. In fact, one can take $\mathcal{A} = \mathcal{B}^a(E)$, where E is some $\mathcal{B}$-correspondence.*

Markov semigroup = unital CP-semigroup.

Bhat's theorem (discrete case) revisited

Theorem

Let T be a Markov semigroup. If the GNS subproduct system of T can be embedded in a **product system**, *then T has a unital dilation $(\mathcal{A}, \vartheta, p)$.*

Theorem

Let T be a UCP map on a C-algebra $\mathcal{B}$. Then there exists a triple $(\mathcal{A}, \vartheta, p)$ such that*

$$T^n(b) = p\vartheta^n(b)p \quad , \quad \textit{for all } n \in \mathbb{N}, b \in \mathcal{B}$$

Proof. We need to show that the GNS subproduct system $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of the semigroup $(T_n := T^n)_{n\in\mathbb{N}}$ embeds into a product system. Define $E_n = \mathcal{E}_1^{\odot n}$. Then $\mathcal{E}_{m+n} \hookrightarrow \mathcal{E}_m \odot \mathcal{E}_n$, by induction:

$$\mathcal{E}_n \hookrightarrow \mathcal{E}_{n-1} \odot \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_1^{\odot n} = E_n$$

preserves structure! By the theorem above, T has a dilation.

An application

Theorem (S.-Skeide, see also Bhat 98, Solel 2006)

Every Markov semigroup over $\mathbb{N}^2$ has a unital dilation:
*If T_1, T_2 are two commuting normal unital CP maps on a vN algebra $\mathcal{B}$, then there exist two commuting normal unital *-endomorphisms ϑ_1, ϑ_2 on a vN algebra $\mathcal{A}$ containing $\mathcal{B}$, a projection $p \in \mathcal{A}$ such that $\mathcal{B} = p\mathcal{A}p$, and*

$$T_1^{n_1} \circ T_2^{n_2}(b) = p\vartheta_1^{n_1} \circ \vartheta_2^{n_2}(b)p \quad \textit{for all } b \in \mathcal{B}, n_1, n_2 \in \mathbb{N}.$$

Proof.

Given a Markov semigroup over $\mathbb{N}^2$, we construct a product system that contains the GNS subproduct system of that semigroup. Then apply previous theorem. □

The converse direction

A sufficient condition for the existence of a dilation for a unital CP-semigroup T is that its GNS subproduct system embeds into a product system.

What about the converse direction?

Theorem (S.-Skeide)

• *If a normal Markov semigroup* $T = (T_s)_{s \in \mathbb{S}}$ *has a* **minimal** *normal dilation then its GNS subproduct system embeds into a product system.*

• *A Markov semigroup* $T = (T_s)_{s \in \mathbb{S}}$ *has a* **strict** *dilation* $(\mathcal{B}^a(E), \vartheta, p)$ *where* E *is a* $\mathcal{B}$*-correspondence,* **if and only if** *its GNS subproduct system embeds into a product system.*

1. We did not define what "minimal" means.
2. Over $\mathbb{N}^k$ $(k \geq 2)$, minimal dilations are not unique.
3. Over $\mathbb{N}^k$ $(k \geq 2)$, a given dilation might not be "minimalizable", that is, cannot be compressed or restricted to a minimal one (new and weird).
4. What about dilations $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A} \neq \mathcal{B}^a(E)$?

The converse direction II

Theorem (S.-Skeide)

• *If a normal Markov semigroup* $T = (T_s)_{s \in \mathbb{S}}$ *has a normal* **minimal** *dilation then its GNS subproduct system embeds into a product system.*

Corollary (S.-Skeide)

There exist CP and Markov semigroups over $\mathbb{N}^3$ *for which there is no* **minimal** *dilation.*

"Proof" (not really...)

[S.-Solel] construct a subproduct system over $\mathbb{N}^3$ that cannot be embedded into a product system. We apply the above theorem to that subproduct system.

Problem: this does not rule out the existence of **non-minimal** dilations.

Minimality, von Neumann case

Let $T = (T_s)_{s\in\mathbb{S}}$ be a CP-semigroup over $\mathbb{S}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. Suppose that $\mathcal{B} \subseteq \mathcal{B}(H)$ and that $\mathcal{A} \subseteq \mathcal{B}(K)$, so that $p = P_H$. There are three properties that one may require for "minimality":

1. "Algebraic minimality", that is
$$\mathcal{A} = W^*(\cup_{s\in\mathbb{S}}\vartheta_s(\mathcal{B})).$$
2. "Spatial minimality", that is, $\mathcal{A} = \overline{\mathcal{A}p\mathcal{A}}^s$. Assuming 1, same as:
$$K = \overline{\mathrm{span}\{\vartheta_{s_1}(b_1)\cdots\vartheta_{s_n}(b_n)h : s_i \in \mathbb{S}, b_i \in \mathcal{B}, h \in H\}}.$$
3. "Incompressibility": there is no nontrivial projection $p \leq q \in \mathcal{A}$ s.t.
$$q\vartheta_s(\cdot)q : q\mathcal{A}q \to q\mathcal{A}q \quad , \quad q\vartheta_s(\cdot)q : qaq \mapsto q\vartheta_s(qaq)q,$$
is an E-semigroup, and a dilation of T.

Minimality, von Neumann case (cont.)

1. $\mathcal{A} = W^*(\cup_{s\in\mathbb{S}}\vartheta_s(\mathcal{B}))$.
2. $\mathcal{A} = \overline{\mathcal{A}p\mathcal{A}}^s$.
3. No nontrivial projection $p \leq q \neq 1$ in $\mathcal{A}$ s.t. $q\vartheta_s(\cdot)q$ is a dilation.

The notion of **minimality** referred to in theorem and corollary above is the strongest one: 1+2. (This also implies 3).

It is easy to restrict to a semigroup satisfying 1, and not hard to compress to obtain 1+3, but that is not the notion that works best.

Over $\mathbb{R}_+$ (and $\mathbb{N}$), 1+2 is equivalent to 1+3. (non-trivial!)

We have an example of a dilation $(\mathcal{A}, \vartheta, p)$ over $\mathbb{N}^2$, which satisfies 2, but not 1. After restricting to $W^*(\cup_{s\in\mathbb{S}}\vartheta_s(\mathcal{B}))$, and then compressing to the minimal compressing q, one obtains an algebraically minimal and incompressible dilation (1+3), which does **not** satisfy 2.

Dilation $\Rightarrow$ what?

Let $T = (T_s)_{s\in\mathbb{S}}$ be a Markov semigroup on $\mathcal{B}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. Following a construction from [Skeide02], we see what structure arises. Define a family $(E_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences as follows:

$$E := \mathcal{A}p \quad , \quad E_s := \vartheta_s(p)E.$$

C*-correspondence structure:

$$b \cdot x_s := \vartheta_s(b)x_s \quad , \quad x_s \cdot b := xb, \quad x_s \in E_s, b \in \mathcal{B}.$$

$$\langle x_s, y_s \rangle := x_s^* y_s \in p\mathcal{A}p = \mathcal{B}.$$

Unit:

$$\eta_s := \vartheta_s(p)p \in E_s.$$

(E_s, η_s) represents T

$$\langle \eta_s, b \cdot \eta_s \rangle = p\vartheta_s(p)\vartheta_s(b)\vartheta_s(p)p = p\vartheta_s(b)p = T_s(b).$$

Dilation $\Rightarrow$ what? II

Let $T = (T_s)_{s \in \mathbb{S}}$ be a Markov semigroup on $\mathcal{B}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. We constructed a family $(E_s)_{s \in \mathbb{S}}$ of $\mathcal{B}$-corresopndences, and a family $(\eta_s)_{s \in \mathbb{S}}$ of unit vectors ($\eta_s \in E_s$) that represent T:

$$\langle \eta_s, b \cdot \eta_s \rangle = p \vartheta_s(b) p = T_s(b).$$

Hence (E_s, η_s) "contains" the GNS representation $(\mathcal{E}_s, \xi_s)$ of T_s.

Q: is $(E_s)_{s \in \mathbb{S}}$ a **PRODUCT** system?

Dilation ⇒ what? III

Let $T = (T_s)_{s\in\mathbb{S}}$ be a CP-semigroup on $\mathcal{B}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. Let $((E_s)_{s\in\mathbb{S}}, (\eta_s)_{s\in\mathbb{S}})$ be as above, $\langle \eta_s, b \cdot \eta_s \rangle = T_s(b)$.
Define

$$v_{s,t} : E_s \odot E_t \to E_{s+t}$$

$$v_{s,t} : x_s \odot y_t \mapsto \vartheta_t(x_s) y_t$$

A direct calculation shows:

$$\langle x_s \odot y_t, x'_s \odot y'_t \rangle = \ldots = \langle \vartheta_t(x_s) y_t, \vartheta_t(x'_s) y'_t \rangle.$$

Hence $v_{s,t} : E_s \odot E_t \to E_{s+t}$ is an isometry:

$$E_s \odot E_t \subseteq E_{s+t}.$$

$(E_s)_{s\in\mathbb{S}}$ is a **superproduct system** (but not always a product system).

Superproduct systems[2]

Definition

A **superproduct system** is a family $E^{\otimes} = (E_s)_{s \in \mathbb{S}}$ of $\mathcal{B}$-correspondences, together with a family $\{v_{s,t} : E_s \odot E_t \to E_{s+t}\}$ of isometric bimodule maps, which iterate associatively, i.e., the following diagram is commutative $(\forall r, s, t)$:

$$\begin{array}{ccc} E_r \odot E_s \odot E_t & \longrightarrow & E_r \odot E_{s+t} \\ \downarrow & & \downarrow \\ E_{r+s} \odot E_t & \longrightarrow & E_{r+s+t} \end{array}$$

A **product system** is a superproduct system in which $v_{s,t}$ are all unitaries.

[2] The notion is due to Margetts and Srinivasan

Recap

Subproduct system: $\mathcal{E}_s \odot \mathcal{E}_t \supseteq \mathcal{E}_{s+t}$
Product system: $E_s \odot E_t = E_{s+t}$
Unit: $\xi_s \odot \xi_t = \xi_{s+t}$
Superproduct system: $E_s \odot E_t \subseteq E_{s+t}$

For every CP-semigroup T on $\mathcal{B}$, there exists a subproduct system $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences (the **GNS subproduct system**) and a unit $(\xi_s)_{s\in\mathbb{S}}$ such that

$$T_s(b) = \langle \xi_s, b\xi_s \rangle \quad \text{for all } s \in \mathbb{S}, b \in \mathcal{B}.$$

If T unital, and if the GNS subproduct system can be **embedded into a product system**, then T has a dilation $(\mathcal{A}, \vartheta, p)$ (with $\mathcal{A} = \mathcal{B}^a(E)$).

If T has a dilation $(\mathcal{A}, \vartheta, p)$, then the GNS subproduct system must **embed into a superproduct system.**

Dilations and superproduct systems

Theorem (S.-Skeide)

Let $T = (T_s)_{s \in \mathbb{S}}$ be a Markov semigroup on a von Neumann algebra $\mathcal{B}$.

- *A sufficient condition for T to a have a dilation, is that the GNS subproduct system of T embeds into a* **product** *system.*
- *A necessary condition for T to have a dilation, is that the GNS subproduct system of T embeds into a* **superproduct** *system.*

Corollary (S.-Skeide)

There exist CP and Markov semigroups over $\mathbb{N}^3$ that have **no** *dilation.*

"Proof" (not really...)

We have an example of a subproduct system over $\mathbb{N}^3$ that cannot be embedded into a **superproduct** system.

The truth: the SPS is not the GNS subproduct system of a CP-semigroup, so the proof does not really go like that ...

Another way subproduct systems arise (W* case)

Let E be a full W*-correspondence over $\mathcal{B}$, and $\mathscr{B}^a(E)$ the **adjointable** operators on E. E is a **Morita W* equivalence** from $\mathscr{B}^a(E)$ to $\mathcal{B}$:

$$\mathcal{B} = E^* \overline{\odot}^s E \quad , \quad \mathscr{B}^a(E) = E \overline{\odot}^s E^*.$$

For $T = (T_s)_{s \in \mathbb{S}}$ a CP-s.g. on $\mathscr{B}^a(E)$, and $\mathcal{E}^{\circleddash} = (\mathcal{E}_s)_{s \in \mathbb{S}}$ the GNS SPS consider the **Morita equivalent** subproduct system $\mathcal{F}^{\circleddash} = (\mathcal{F}_s)_{s \in \mathbb{S}}$ given by

$$\mathcal{F}_s := E^* \overline{\odot}^s \mathcal{E}_s \overline{\odot}^s E.$$

$\mathcal{F}^{\circleddash}$ **the subproduct system of $\mathcal{B}$-correspondences associated with T.**

Theorem (S.-Skeide, see also S.-Solel)

Every subproduct system over $\mathcal{B}$ is the subproduct system of $\mathcal{B}$-correspondences associated with some normal CP-semigroup T acting on some $\mathscr{B}^a(E)$, where E is a $\mathcal{B}$-correspondence.
In particular, every SPS is Morita equivalent to the GNS of some CP-semigroup.

Morita equivalence behaves nicely w.r.t. inclusions into product systems.

Thank you!