Dilation theory of CP maps and CP-semigroups

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Technion

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(ii) My blog *Noncommutative Analysis* ->Lecture Notes -> “Topics in Operator Theory 106435". Starts similarly and continues in the operator algebraic direction.

(iii) My forthcoming “paper" with Michael Skeide *CP-Semigroups and Dilations, Subproduct Systems and Superproduct Systems: The Multiparameter Case and Beyond*. 
Completely positive maps

An operator system is a selfadjoint unital subspace $1 \in X \subseteq \mathcal{B}(H)$. A map $\phi : X \to Y$ is said to be positive if $X \ni a \geq 0 \implies \phi(a) \geq 0$. It is completely positive if

$$\phi^{(n)} := \phi \otimes \text{id}_{M_n} : M_n(X) = X \otimes M_n \to M_n(Y) = Y \otimes M_n$$

is positive for all $n \in \mathbb{N}$.

Examples of CP maps

- A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between C*-algebras is CP: $\pi(a^*a) = \pi(a)^*\pi(a)$ so positive, but $\pi^{(n)}$ is also a $*$-homomorphism.
- If $V \in \mathcal{B}(H,K)$ then $\phi : T \mapsto V^*TV$ is completely positive:

$$\phi^{(n)} : (T_{ij}) \mapsto \text{diag}(V^*,\ldots,V^*)(T_{ij}) \text{ diag}(V,\ldots,V)$$

- Sums of CPs, compositions of CPs, etc.
Stinespring’s dilation theorem

Theorem (Stinespring, 1955)

Let $A$ be a unital C*-algebra and let $\phi : A \to \mathcal{B}(H)$ be a CP map. Then there exists a Hilbert space $K$, an operator $V \in \mathcal{B}(H, K)$, and a (unital) $\ast$-representation $\pi : A \to \mathcal{B}(K)$, such that

$$\phi(a) = V^* \pi(a) V \quad , \quad \text{for all } a \in A$$

Moreover, $(\pi, K, V)$ can be chosen minimal, in the sense that $K = [\pi(A)V H]$.  

(i) The minimal dilation above is called the **minimal Stinespring dilation**, and it is unique.

(ii) If $\phi$ is unital then $V^* V = V^* \pi(1) V = \phi(1) = I_H$, so $V$ is an isometry. Then $H \equiv VH$ and write

$$\phi(a) = P_H \pi(a) \big|_H$$
Proof of Stinespring’s dilation theorem

**Theorem (Stinespring, 1955)**

Let $A$ be a unital C*-algebra and let $\phi : A \to \mathcal{B}(H)$ be a CP map. Then there exists a Hilbert space $K$, an operator $V \in \mathcal{B}(H,K)$, and a (unital) $\ast$-representation $\pi : A \to \mathcal{B}(K)$, such that

$$\phi(a) = V^* \pi(a)V \quad , \quad \text{for all } a \in A$$

**Proof.** Construct the Hilbert space $A \otimes_\phi H$ obtained from $A \otimes_{\text{alg}} H$ with the (semi-)inner product (here is where we use CP)

$$\langle a \otimes g, b \otimes h \rangle = \langle g, \phi(a^* b) h \rangle$$

Define $\pi(a)b \otimes h = ab \otimes h$ and $Vh = 1 \otimes h$ (so $V^*(a \otimes h) = \phi(a)h$).

$$V^* \pi(a)Vh = V^* \pi(a)1 \otimes h = V^*(a \otimes h) = \phi(a)h$$

as required.
Application: representation of CP maps on $\mathcal{B}(H)$

**Theorem (Choi-Kraus representation)**

Let $\phi : M_n \rightarrow M_k$ be a CP map. Then there exist $W_i \in M_{n,k}$ such that

$$\phi(A) = \sum W_i A W_i^*, \quad A \in M_n$$

(True also for normal CP maps on $\mathcal{B}(H)$).

**Proof.** $\phi(\cdot) = V^* \pi(\cdot) V$. By basic representation theory of $M_n$:

$$\pi(A) \cong A \oplus A \oplus \cdots \oplus A = \sum V_i AV_i^*$$

where $V_i : \mathbb{C}^n \rightarrow \mathbb{C}^{mn}$ the isometry into the $i$th summand in $\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n$. Put $W_i = V^* V_i$.

$$\phi(A) = V^* (\sum V_i AV_i^*) V = \sum W_i A W_i^*$$
Application: a dilation machine
Example: vN inequality ⇒ unitary dilation

Suppose that \( \|T\| \leq 1 \), and that we know

\[
\|p(T)\| \leq \|p\|_\infty \equiv \sup_{|z|=1} |p(z)|
\]

\( C(\mathbb{T}) \supset \mathbb{C}[z] \ni p \mapsto p(T) \in B(H) \) is unital and contractive.

\[
\implies C(\mathbb{T}) \supset \mathbb{C}[z] + \overline{\mathbb{C}[z]} \ni p + q \mapsto p(T) + q(T)^* \in B(H)
\]

is unital and positive. \( \mathbb{C}[z] + \overline{\mathbb{C}[z]} \) is dense in \( C(\mathbb{T}) \) ⇒ we obtain a unital positive \( \phi : C(\mathbb{T}) \to B(H) \) s.t. \( \phi(p) = p(T) \) for polynomials.

\( C(\mathbb{T}) \) is commutative ⇒ \( \phi \) is UCP. **Stinespring**: \( \pi : C(\mathbb{T}) \to B(K) \)

\[
p(T) = \phi(p) = P_H \pi(p) \big|_H = P_H \phi(p(\pi(z))) \big|_H = P_H \phi(p(U)) \big|_H
\]

\( U = \pi(z) \) is unitary because \( \pi \) is a \(*\)-homomorphism and \( z \) is unitary.
A dilation machine

Example: Ando inequality \(\Rightarrow\) Ando dilation?

Suppose that \(\|T_1\|, \|T_2\| \leq 1\), and that we know

\[
\|p(T_1, T_2)\| \leq \|p\|_\infty := \sup_{|z_1|=|z_2|=1} |p(z_1, z_2)|
\]

\(C(\mathbb{T}^2) \supset \mathbb{C}[z_1, z_2] \ni p \mapsto p(T_1, T_2) \in \mathcal{B}(H)\) is unital and contractive.

\[
\implies \mathbb{C}[z_1, z_2] + \overline{\mathbb{C}[z_1, z_2]} \ni p + \overline{q} \mapsto p(T_1, T_2) + q(T_1, T_2)^* \in \mathcal{B}(H)
\]
is unital and positive. \(\mathbb{C}[z_1, z_2] + \overline{\mathbb{C}[z_1, z_2]}\) is dense in \(C(\mathbb{T}^2)\)? No! The argument breaks down. Its true that the map is UCP, but this doesn’t help. We need \(\phi\) to be defined on a \(C^*\)-algebra to use Stinespring’s theorem. If we can extend \(p \mapsto p(T_1, T_2)\) to a UCP map \(\phi : C(\mathbb{T}^2) \to \mathcal{B}(H)\), then we can apply Stinespring as before:

\[
p(T_1, T_2) = \phi(p) = P_H \pi(p)|_H = P_H p(\pi(z_1), \pi(z_2))|_H = P_H p(U_1, U_2)|_H
\]

\(U_i = \pi(z_i)\) is unitary.
Arveson’s extension theorem and C*-dilations

**Theorem (Arveson, 1969)**

Let $X \subset A$ be an operator system contained in a C*-algebra $B$. Let $\phi : X \to B(H)$ be a CP map. Then there exists a CP map $\tilde{\phi} : B \to B(H)$ such that $\|\tilde{\phi}\| = \|\phi\|$ and which extends $\phi$: $\tilde{\phi}(x) = \phi(x)$ for all $x \in X$.

For a proof, see Paulsen’s book (fails for positive). Arveson’s theorem can be used together with Stinespring’s theorem to obtain dilation theorems.

**Definition**

Let $1 \in X \subseteq B$ be a unital operator space. A linear map $\phi : X \to B(H)$ is said to have a C*-dilation to $B$ if there exists a $*$-representation $\pi : B \to B(K)$, $K \supseteq H$, such that

$$\phi(x) = P_H \pi(x)\big|_H,$$

for all $x \in X$.

**Theorem (Arveson, 1969)**

Every UCP (or UCC) map has a C*-dilation.
A tuple $T = (T_1, \ldots, T_d)$ is said to be a **row contraction** if $\sum T_i T_i^* \leq I$. It is a **row isometry** if $\sum T_i^* T_j = \delta_{ij} I$.

**Theorem**

*Every row contraction has a row isometric dilation.*

By this we mean a row isometry $(V_i)$ on $\mathcal{B}(K), K \supset H$, such that

$$T^\alpha = T_{\alpha_1} \cdots T_{\alpha_k} = P_H V_{\alpha_1} \cdots V_{\alpha_k} |_{H} = P_H V^\alpha |_{H}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, 2, \ldots, d\}^k$, for all $k$. 
**Example: row isometric dilation**

**Theorem**

*Every row contraction has a row isometric dilation.*

**Proof** ($d = 1$). Let $T \in \mathcal{B}(H)$, $\|T\| \leq 1$. Let $S$ be the unilateral shift on $\ell^2$. For $r \in (0, 1)$, let $D_rT = (I - r^2TT^*)^{1/2}$, and define $K_r(T) : H \to \ell^2 \otimes H$ by

$$K_r(T)h = \sum_n e_n \otimes (r^n D_rT T^n h)$$

$$K_r(T)^* K_r(T)h = \sum r^{2n} T^n D_r^2 T^n h = \sum r^{2n} T^n (I - r^2 T T^*) T^n h =$$

$$= \sum r^{2n} T^n T^n h - \sum r^{2(n+1)} T^{n+1} T^{(n+1)*} h = h$$

$$K_r(T)^* (S \otimes I) K_r(T)h = K_r(T)^* \sum e_{n+1} \otimes (r^n D_r T T^n h) =$$

$$= \sum r^{2n+1} T^{n+1} D_r^2 T T^n h = r Th$$
Example: row isometric dilation

Proof continued

Let $T \in \mathcal{B}(H)$, $\|T\| \leq 1$. Let $S$ be the unilateral shift on $\ell^2$. On $C^*(S)$ we define a CP map

$$\phi_r(a) = K_r(T)^* (a \otimes I) K_r(T)$$

We saw: $\phi_r(I) = I$, $\phi_r(S) = rT$. Likewise, $\phi_r(S^n) = r^n T^n$. Define a UCP $\Phi := \lim_{r \to 1} \phi_r$

$$\Phi(S^n) = T^n$$

Let $\pi : \mathcal{T} \rightarrow \mathcal{B}(K)$ be a $C^*$-dilation of $\Phi$. Then

$$T^n = \Phi(S^n) = P_H\pi(S^n)|_H = P_HV^n|_H$$

where $V = \pi(S)$ is an isometry, being the image of an isometry under a (unital) $*$-homomorphism.
Dilation theory of completely positive semigroups
The objects of study

$S$ a semigroup of $\mathbb{R}_+^k$, such that $0 \in S$.

$T = (T_s)_{s \in S}$ a family of maps on a unital C*-algebra $\mathcal{B}$.

- $T$ is said to be a **CP-semigroup** (over $S$) if
  1. $T_s$ is a (contractive) CP map for all $s$,
  2. $T_0 = \text{id}_\mathcal{B}$,
  3. $T_{s+t} = T_s \circ T_t$, for all $s, t \in S$.

- If $T_s(1) = 1$ for all $s$, then $T$ is said to be a **Markov semigroup**.
- If $T_s$ is a $*$-endomorphism for all $s$, then $T$ is said to be an **E-semigroup**.
- Case of greatest interest: $S = \mathbb{R}_+$, then CP-semigroups $T = (T_t)_{t \geq 0}$ (and E-semigroups) have quantum dynamical interpretations.

\[(\text{UCP}) \quad t \mapsto T_t(a) \quad \text{evolution in an irreversible quantum system}\]

\[(\text{*auto}) \quad t \mapsto \alpha_t(a) \quad \text{evolution in a reversible quantum system}\]

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The objects of study II

\[ 0 \in S \subseteq \mathbb{R}_+^k. \]

\[ T = (T_s)_{s \in S} \text{ a CP-semigroup on a unital C*-algebra } \mathcal{B}. \]

**Example**

If \( T_1, \ldots, T_k \) are \( k \) commuting CP maps, then we get a CP-semigroup \((T_s)_{s \in \mathbb{N}^k}\) over \( S = \mathbb{N}^k \):

\[ T_s = T_1^{s_1} \circ \cdots \circ T_k^{s_k} \quad \text{where} \quad s = (s_1, \ldots, s_k) \in \mathbb{N}^k. \]

Every CP-semigroup over \( S = \mathbb{N}^k \) arises this way.

**Issue:** The Stinespring dilations of different \( T_s \) do not work well together.
Bhat’s dilation theorem

Theorem (Bhat, 1996)

Let \( T = (T_t)_{t \geq 0} \) be a CP-semigroup on \( \mathcal{B}(H) \). Then there exists a Hilbert space \( K \) containing \( H \), and an \( E \)-semigroup \( \vartheta = (\vartheta_t)_{t \geq 0} \) on \( \mathcal{B}(K) \), such that

\[
T_t(A) = P_H \vartheta_t(A) P_H, \quad \text{for all } t \geq 0 \text{ and } A \in \mathcal{B}(H).
\]
Bhat’s dilation theorem

**Theorem (Bhat, 1996)**

Let $T = (T_t)_{t \geq 0}$ be a CP-semigroup on $\mathcal{B}(H)$. Then there exists a Hilbert space $K$ containing $H$, and an E-semigroup $\varrho = (\varrho_t)_{t \geq 0}$ on $\mathcal{B}(K)$, such that

$$T_t(A) = P_H \varrho_t(A) P_H \quad , \quad \text{for all } t \geq 0 \text{ and } A \in \mathcal{B}(H).$$

**Interpretation**

An irreversible quantum dynamical system can be embedded in a reversible one ($\varrho$ can be extended to a group of *-automorphisms).

**Application**

An index for quantum dynamical semigroups (Bhat).

Other notions of dilations of CP-semigroups have been studied since 70s: Davies, Evans-Lewis, Hudson-Parthasarathy, Kummerer, Sauvageout ...
Bhat’s theorem – discrete case (toy version)

**Theorem**

Let $T$ be a normal CCP map on $\mathcal{B}(H)$. Then there exists a Hilbert space $K$ containing $H$, and a normal $*$-endomorphism $\vartheta$ on $\mathcal{B}(K)$, such that

$$T^n(A) = P_H \vartheta^n(A) P_H, \quad \text{for all } n \in \mathbb{N} \text{ and } A \in \mathcal{B}(H).$$

**Proof.** We know that $T(A) = \sum W_i A W_i^*$. Assume $T(A) = W A W^*$. $W W^* = T(I) \leq I$ ($T$ is contractive), so $W$ is a contraction. Let $V \in \mathcal{B}(K)$ be an isometric dilation of $W$ define

$$\vartheta(B) = VBV^*, \quad B \in \mathcal{B}(K)$$

This is an endomorphism:

$$\vartheta(B_1)\vartheta(B_2) = VB_1V^*VB_2V^* = VB_1B_2V^* = \vartheta(B_1B_2)$$

For $A = P_H A P_H \in \mathcal{B}(H)$,

$$P_H \vartheta^n(A) P_H = P_H V^n P_H A P_H V^n P_H = W^n A W^n = T^n(A)$$
Let $T$ be a normal CCP map on $\mathcal{B}(H)$. Then there exists a Hilbert space $K$ containing $H$, and a normal $*$-endomorphism $\vartheta$ on $\mathcal{B}(K)$, such that

$$T^n(A) = P_H \vartheta^n(A) P_H, \quad \text{for all } n \in \mathbb{N} \text{ and } A \in \mathcal{B}(H).$$

**Proof.** We know that $T(A) = \sum W_i A W_i^*$. 
$\sum W_i W_i^* = T(I) \leq I$, so $W = (W_i)$ is a row contraction. 
Let $V_i \in \mathcal{B}(K)$ be a row isometric dilation of $(W_i)$ define

$$\vartheta(B) = \sum V_i B V_i^*, \quad B \in \mathcal{B}(K)$$

This is an endomorphism (recall $V_i^* V_j = \delta_{ij} I_K$):

$$\vartheta(B_1) \vartheta(B_2) = \sum V_i B_1 V_i^* \sum V_j B_2 V_j^* = \sum V_i B_1 B_2 V_i^* = \vartheta(B_1 B_2)$$

$$P_H \vartheta^n(A) P_H = \sum_{|\alpha| = n} P_H V^\alpha P_H A P_H V^{\alpha^*} P_H = \sum_{|\alpha| = n} W^\alpha A W^{\alpha^*} = T^n(A)$$
We study the possible generalizations of Bhat’s theorem to a CP-semigroup $T$ on a unital C*-algebra $\mathcal{B}$, parameterized by a semigroup $S \subseteq \mathbb{R}^k_+$. 

**Definition**

A **dilation** of $T$ is a triple $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A}$ is a C*-algebra, $\vartheta = (\vartheta_s)_{s \in S}$ is a semigroup of *-endomorphisms, and $p \in \mathcal{A}$ is a projection, such that $\mathcal{B} = p\mathcal{A}p$, and such that

$$T_s(b) = p\vartheta_s(b)p \quad \text{for all } b \in \mathcal{B}, s \in S.$$
We study the possible generalizations of Bhat’s theorem to a CP-semigroup $T$ on a unital C*-algebra $B$, paramaterized by a semigroup $S \subseteq \mathbb{R}_+^k$.

**Definition**

A **dilation** of $T$ is a triple $(A, \vartheta, p)$, where $A$ is a C*-algebra, $\vartheta = (\vartheta_s)_{s \in S}$ is a semigroup of *-endomorphisms, and $p \in A$ is a projection, such that $B = pAp$, and such that

$$T_s(b) = p\vartheta_s(b)p \quad \text{for all } b \in B, s \in S.$$  

**Questions**

1. Find necessary & sufficient conditions for existence of dilation.
2. For fixed $k$, does every CP-semigroup over $\mathbb{N}^k$ have a dilation?
Key tool: C*-correspondences

Let $B$ be a C*-algebra. A **Hilbert C*-moudle** over $B$ is a right module $E$ that has a $B$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to B$, such that

(i) $\langle x, x \rangle \geq 0$ for all $x \in E$,

(ii) $\langle x, yb \rangle = \langle x, y \rangle b$ for all $x, y \in E$ and $b \in B$,

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in E$,

(iv) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$,

(v) $\|x\| := \|\langle x, x \rangle\|^{1/2}$ is a norm on $E$ which makes $E$ into a Banach space.

A **C*-correspondence** is a Hilbert C*-module that also has a left action by adjointable operators.

The tensor product $E \otimes F$: obtained from $E \otimes_{alg} F$ by inner product

$$\langle x \otimes y, x' \otimes y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$$
The GNS representation \((\mathcal{E}, \xi)\) of a CP map

Let \(T : \mathcal{B} \to \mathcal{B}\) be a CP map. Then there exists a unique \(C^*\)-correspondence \(\mathcal{E}\) over \(\mathcal{B}\), and a vector \(\xi \in \mathcal{E}\), such that

\[
\text{span} \overline{\mathcal{B}\xi}\mathcal{B} = \mathcal{E}
\]

and

\[
\langle \xi, b\xi \rangle = T(b) \quad \text{for all } b \in \mathcal{B}.
\]

**Construction:** on \(\mathcal{E}_0 = \mathcal{B} \otimes_{\text{alg}} \mathcal{B}\) put inner product

\[
\langle a \otimes b, c \otimes d \rangle = b^* T(a^* c)d
\]

and bimodule operation

\[
a(x \otimes y)d = ax \otimes yd.
\]

**Complete** the quotient, and put \(\xi = 1 \otimes 1\). This works:

\[
\langle \xi, b\xi \rangle = \langle 1 \otimes 1, b \otimes 1 \rangle = 1^* T(1^* b)1 = T(b).
\]
The GNS representation \((\mathcal{E}, \xi)\) of a CP map

Let \(T : \mathcal{B} \to \mathcal{B}\) be a CP map. Then there exists a unique \(C^*\)-correspondence \(\mathcal{E}\) over \(\mathcal{B}\), and a vector \(\xi \in \mathcal{E}\), such that

\[
\text{span} \overline{\mathcal{B}\xi\mathcal{B}} = \mathcal{E}
\]

and

\[
\langle \xi, b\xi \rangle = T(b) \quad \text{for all } b \in \mathcal{B}.
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**Construction:** on \(\mathcal{E}_0 = \mathcal{B} \otimes_{\text{alg}} \mathcal{B}\) put inner product

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and bimodule operation

\[
a(x \otimes y)d = ax \otimes yd.
\]

**Complete** the quotient, and put \(\xi = 1 \otimes 1\). This works:

\[
\langle \xi, b\xi \rangle = \langle 1 \otimes 1, b \otimes 1 \rangle = 1^* T(1^* b)1 = T(b).
\]
The GNS representation of a CP-semigroup

Let \( T = (T_s)_{s \in S} \) be a CP-semigroup on \( \mathcal{B} \).

For every \( s \), let \((\mathcal{E}_s, \xi_s)\) be the GNS representation of \( T_s \).

For \( s, t \in S \), define

\[
w_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \otimes \mathcal{E}_t
\]

by

\[
w_{s,t} : a\xi_{s+t}b \mapsto a\xi_s \otimes \xi_t b,
\]

and then extend linearly. We check:

\[
\langle a\xi_s \otimes \xi_t b, a\xi_s \otimes \xi_t b \rangle = \langle \xi_t b, \langle a\xi_s, a\xi_s \rangle \xi_t b \rangle = b^* \langle \xi_t, T_s(a^*a) \xi_t \rangle b =
\]

\[
= b^* T_t(T_s(a^*a))b = b^* T_{t+s}(a^*a)b = \langle a\xi_{s+t}b, a\xi_{s+t}b \rangle.
\]

\( w_{s,t} \) is an isometry!
Subproduct systems

A **subproduct system** is a family $\mathcal{E}^\otimes = (\mathcal{E}_s)_{s \in S}$ of $\mathcal{B}$-correspondences, together with a family $\{w_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \otimes \mathcal{E}_t\}$ of isometric bimodule maps, which iterate associatively, i.e., the following diagram is commutative $(\forall r, s, t)$:

$$
\begin{array}{ccc}
\mathcal{E}_{r+s+t} & \longrightarrow & \mathcal{E}_r \otimes \mathcal{E}_{s+t} \\
\downarrow & & \downarrow \\
\mathcal{E}_{r+s} \otimes \mathcal{E}_t & \longrightarrow & \mathcal{E}_r \otimes \mathcal{E}_s \otimes \mathcal{E}_t
\end{array}
$$

A **product system** is a subproduct system in which $w_{s,t}$ are all unitaries.

**Definition**

A family $\{\xi_s \in \mathcal{E}_s\}_{s \in S}$ is called a **unit** if $w_{s,t} \xi_{s+t} = \xi_s \otimes \xi_t$ for all $s, t$.

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1 Inclusion systems by Bhat-Mukherjee; recall the talk by Vijay Kumar U. Introduced also by S.-Solel.
Recap

Subproduct system: \( \mathcal{E}_s \odot \mathcal{E}_t \supseteq \mathcal{E}_{s+t} \)
Product system: \( E_s \odot E_t = E_{s+t} \)
Unit: \( \xi_s \odot \xi_t = \xi_{s+t} \)

For every CP-semigroup on \( \mathcal{B} \), there exists a subproduct system \( \mathcal{E}^\otimes = (\mathcal{E}_s)_{s \in \mathbb{S}} \) of \( \mathcal{B} \)-correspondences (called the GNS subproduct system) and a unit \( (\xi_s)_{s \in \mathbb{S}} \) such that

\[
T_s(b) = \langle \xi_s, b\xi_s \rangle \quad \text{for all } s \in \mathbb{S}, b \in \mathcal{B}.
\]

Theorem (S.-Skeide, following Bhat-Skeide, 2000)

Let \( T \) be a Markov semigroup. If the GNS subproduct system of \( T \) can be embedded in a product system, then \( T \) has a unital dilation \((\mathcal{A}, \vartheta, p)\). In fact, one can take \( \mathcal{A} = \mathcal{B}^a(E) \), where \( E \) is some \( \mathcal{B} \)-correspondence.

Markov semigroup = unital CP-semigroup.
Bhat’s theorem (discrete case) revisited

**Theorem**

Let $T$ be a Markov semigroup. If the GNS subproduct system of $T$ can be embedded in a **product system**, then $T$ has a unital dilation $(\mathcal{A}, \vartheta, p)$.

**Theorem**

Let $T$ be a UCP map on a $C^*$-algebra $\mathcal{B}$. Then there exists a triple $(\mathcal{A}, \vartheta, p)$ such that

$$T^n(b) = p\vartheta^n(b)p \quad , \quad \text{for all } n \in \mathbb{N}, b \in \mathcal{B}$$

**Proof.** We need to show that the GNS subproduct system $(\mathcal{E}_n)_{n \in \mathbb{N}}$ of the semigroup $(T_n := T^n)_{n \in \mathbb{N}}$ embeds into a product system. Define $E_n = \mathcal{E}_1^\otimes n$. Then $\mathcal{E}_{m+n} \hookrightarrow \mathcal{E}_m \otimes \mathcal{E}_n$, by induction:

$$\mathcal{E}_n \hookrightarrow \mathcal{E}_{n-1} \otimes \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_1^\otimes n = E_n$$

preserves structure! By the theorem above, $T$ has a dilation.
Theorem (S.-Skeide, see also Bhat 98, Solel 2006)

Every Markov semigroup over $\mathbb{N}^2$ has a unital dilation:

If $T_1, T_2$ are two commuting normal unital CP maps on a vN algebra $\mathcal{B}$, then there exist two commuting normal unital *-endomorphisms $\vartheta_1, \vartheta_2$ on a vN algebra $\mathcal{A}$ containing $\mathcal{B}$, a projection $p \in \mathcal{A}$ such that $\mathcal{B} = p\mathcal{A}p$, and

$$T_1^{n_1} \circ T_2^{n_2}(b) = p\vartheta_1^{n_1} \circ \vartheta_2^{n_2}(b)p \quad \text{for all } b \in \mathcal{B}, n_1, n_2 \in \mathbb{N}.$$

Proof.

Given a Markov semigroup over $\mathbb{N}^2$, we construct a product system that contains the GNS subproduct system of that semigroup. Then apply previous theorem.
A sufficient condition for the existence of a dilation for a unital CP-semigroup $T$ is that its GNS subproduct system embeds into a product system.

What about the converse direction?

**Theorem (S.-Skeide)**

- If a normal Markov semigroup $T = (T_s)_{s \in S}$ has a minimal normal dilation then its GNS subproduct system embeds into a product system.
- A Markov semigroup $T = (T_s)_{s \in S}$ has a strict dilation $(\mathcal{B}^a(E), \vartheta, p)$ where $E$ is a $\mathcal{B}$-correspondence, if and only if its GNS subproduct system embeds into a product system.

1. We did not define what "minimal" means.
2. Over $\mathbb{N}^k$ ($k \geq 2$), minimal dilations are not unique.
3. Over $\mathbb{N}^k$ ($k \geq 2$), a given dilation might not be "minimalizable", that is, cannot be compressed or restricted to a minimal one (new and weird).
4. What about dilations $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A} \neq \mathcal{B}^a(E)$?
The converse direction II

**Theorem (S.-Skeide)**

- If a normal Markov semigroup \( T = (T_s)_{s \in \mathbb{S}} \) has a normal minimal dilation then its GNS subproduct system embeds into a product system.

**Corollary (S.-Skeide)**

There exist CP and Markov semigroups over \( \mathbb{N}^3 \) for which there is no minimal dilation.

"Proof" (not really...)

[S.-Solel] construct a subproduct system over \( \mathbb{N}^3 \) that cannot be embedded into a product system. We apply the above theorem to that subproduct system.

Problem: this does not rule out the existence of non-minimal dilations.
Minimality, von Neumann case

Let $T = (T_s)_{s \in S}$ be a CP-semigroup over $S$, and $(\mathcal{A}, \vartheta, p)$ a dilation. Suppose that $\mathcal{B} \subseteq \mathcal{B}(H)$ and that $\mathcal{A} \subseteq \mathcal{B}(K)$, so that $p = P_H$.

There are three properties that one may require for "minimality":

1. "Algebraic minimality", that is

   $$\mathcal{A} = W^*(\bigcup_{s \in S} \vartheta_s(\mathcal{B})).$$

2. "Spatial minimality", that is, $\mathcal{A} = \overline{A p A}^s$. Assuming 1, same as:

   $$K = \text{span}\{\vartheta_{s_1}(b_1) \cdots \vartheta_{s_n}(b_n)h : s_i \in S, b_i \in \mathcal{B}, h \in H\}.$$

3. "Incompressibility": there is no nontrivial projection $p \leq q \in \mathcal{A}$ s.t.

   $$q\vartheta_s(\cdot)q : q\mathcal{A}q \to q\mathcal{A}q, \quad q\vartheta_s(\cdot)q : qa \mapsto q\vartheta_s(qaq)q,$$

   is an E-semigroup, and a dilation of $T$. 


Minimality, von Neumann case (cont.)

1. \( A = W^*(\bigcup_{s \in S} \vartheta_s(B)) \).
2. \( A = \overline{ApA}^s \).
3. No nontrivial projection \( p \leq q \neq 1 \) in \( A \) s.t. \( q\vartheta_s(\cdot)q \) is a dilation.

The notion of minimality referred to in theorem and corollary above is the strongest one: 1+2. (This also implies 3).

It is easy to restrict to a semigroup satisfying 1, and not hard to compress to obtain 1+3, but that is not the notion that works best.

Over \( \mathbb{R}_+ \) (and \( \mathbb{N} \)), 1+2 is equivalent to 1+3. (non-trivial!)

We have an example of a dilation \((A, \vartheta, p)\) over \( \mathbb{N}^2 \), which satisfies 2, but not 1. After restricting to \( W^*(\bigcup_{s \in S} \vartheta_s(B)) \), and then compressing to the minimal compressing \( q \), one obtains an algebraically minimal and incompressible dilation (1+3), which does not satisfy 2.
Dilation ⇒ what?

Let $T = (T_s)_{s \in S}$ be a Markov semigroup on $\mathcal{B}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. Following a construction from [Skeide02], we see what structure arises. Define a family $(E_s)_{s \in S}$ of $\mathcal{B}$-correspondences as follows:

$$E := Ap, \quad E_s := \vartheta_s(p)E.$$ 

C*-correspondence structure:

$$b \cdot x_s := \vartheta_s(b)x_s, \quad x_s \cdot b := xb, \quad x_s \in E_s, b \in \mathcal{B}.$$ 

$$\langle x_s, y_s \rangle := x_s^* y_s \in pAp = \mathcal{B}.$$ 

Unit:

$$\eta_s := \vartheta_s(p)p \in E_s.$$ 

$(E_s, \eta_s)$ represents $T$

$$\langle \eta_s, b \cdot \eta_s \rangle = p\vartheta_s(p)\vartheta_s(b)\vartheta_s(p)p = p\vartheta_s(b)p = T_s(b).$$
Let $T = (T_s)_{s \in S}$ be a Markov semigroup on $\mathcal{B}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. We constructed a family $(E_s)_{s \in S}$ of $\mathcal{B}$-correspondences, and a family $(\eta_s)_{s \in S}$ of unit vectors ($\eta_s \in E_s$) that represent $T$:

$$\langle \eta_s, b \cdot \eta_s \rangle = p\vartheta_s(b)p = T_s(b).$$

Hence $(E_s, \eta_s)$ "contains" the GNS representation $(\mathcal{E}_s, \xi_s)$ of $T_s$.

Q: is $(E_s)_{s \in S}$ a PRODUCT system?
Dilation ⇒ what? III

Let \( T = (T_s)_{s \in S} \) be a CP-semigroup on \( \mathcal{B} \), and \((A, \vartheta, p)\) a dilation. Let \(( (E_s)_{s \in S}, (\eta_s)_{s \in S} ) \) be as above, \( \langle \eta_s, b \cdot \eta_s \rangle = T_s(b) \).

Define

\[
\nu_{s,t} : E_s \odot E_t \to E_{s+t}
\]

\[
\nu_{s,t} : x_s \odot y_t \mapsto \vartheta_t(x_s)y_t
\]

A direct calculation shows:

\[
\langle x_s \odot y_t, x'_s \odot y'_t \rangle = \ldots = \langle \vartheta_t(x_s)y_t, \vartheta_t(x'_s)y'_t \rangle.
\]

Hence \( \nu_{s,t} : E_s \odot E_t \to E_{s+t} \) is an isometry:

\[
E_s \odot E_t \subseteq E_{s+t}.
\]

\((E_s)_{s \in S}\) is a superproduct system (but not always a product system).
A superproduct system is a family $E^\otimes = (E_s)_{s \in S}$ of $\mathcal{B}$-correspondences, together with a family $\{v_{s,t} : E_s \odot E_t \to E_{s+t}\}$ of isometric bimodule maps, which iterate associatively, i.e., the following diagram is commutative $(\forall r, s, t)$:

$$
\begin{array}{ccc}
E_r \odot E_s \odot E_t & \longrightarrow & E_r \odot E_{s+t} \\
\downarrow & & \downarrow \\
E_{r+s} \odot E_t & \longrightarrow & E_{r+s+t}
\end{array}
$$

A product system is a superproduct system in which $v_{s,t}$ are all unitaries.

---

2 The notion is due to Margetts and Srinivasan
Recap

Subproduct system: \( \mathcal{E}_s \circ \mathcal{E}_t \supseteq \mathcal{E}_{s+t} \)

Product system: \( E_s \circ E_t = E_{s+t} \)

Unit: \( \xi_s \circ \xi_t = \xi_{s+t} \)

Superproduct system: \( E_s \circ E_t \subseteq E_{s+t} \)

For every CP-semigroup \( T \) on \( \mathcal{B} \), there exists a subproduct system \( \mathcal{E}^\oplus = (\mathcal{E}_s)_{s \in \mathbb{S}} \) of \( \mathcal{B} \)-correspondences (the **GNS subproduct system**) and a unit \( (\xi_s)_{s \in \mathbb{S}} \) such that

\[
T_s(b) = \langle \xi_s, b\xi_s \rangle \quad \text{for all } s \in \mathbb{S}, b \in \mathcal{B}.
\]

If \( T \) unital, and if the GNS subproduct system can be embedded into a product system, then \( T \) has a dilation \( (\mathcal{A}, \vartheta, p) \) (with \( \mathcal{A} = \mathcal{B}^a(E) \)).

If \( T \) has a dilation \( (\mathcal{A}, \vartheta, p) \), then the GNS subproduct system must embed into a superproduct system.
Dilations and superproduct systems

Theorem (S.-Skeide)

Let $T = (T_s)_{s \in S}$ be a Markov semigroup on a von Neumann algebra $\mathcal{B}$.

- A sufficient condition for $T$ to have a dilation, is that the GNS subproduct system of $T$ embeds into a product system.
- A necessary condition for $T$ to have a dilation, is that the GNS subproduct system of $T$ embeds into a superproduct system.

Corollary (S.-Skeide)

There exist CP and Markov semigroups over $\mathbb{N}^3$ that have no dilation.

"Proof" (not really...)

We have an example of a subproduct system over $\mathbb{N}^3$ that cannot be embedded into a superproduct system.

The truth: the SPS is not the GNS subproduct system of a CP-semigroup, so the proof does not really go like that...
Another way subproduct systems arise ($W^*$ case)

Let $E$ be a full $W^*$-correspondence over $B$, and $B^a(E)$ the adjointable operators on $E$. $E$ is a Morita $W^*$ equivalence from $B^a(E)$ to $B$:

$$B = E^{*\@s}E, \quad B^a(E) = E\@s E^*.$$

For $T = (T_s)_{s \in S}$ a CP-s.g. on $B^a(E)$, and $\mathcal{E}^{\otimes} = (\mathcal{E}_s)_{s \in S}$ the GNS SPS consider the Morita equivalent subproduct system $\mathcal{F}^{\otimes} = (\mathcal{F}_s)_{s \in S}$ given by

$$\mathcal{F}_s := E^{*\@s} \mathcal{E}_s \@s E.$$

$\mathcal{F}^{\otimes}$ the subproduct system of $B$-correspondences associated with $T$.

Theorem (S.-Skeide, see also S.-Solel)

Every subproduct system over $B$ is the subproduct system of $B$-correspondences associated with some normal CP-semigroup $T$ acting on some $B^a(E)$, where $E$ is a $B$-correspondence.

In particular, every SPS is Morita equivalent to the GNS of some CP-semigroup.

Morita equivalence behaves nicely w.r.t. inclusions into product systems.
Thank you!