Algebras of bounded analytic nc functions on nc varieties

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Based on a joint work with **Guy Salomon** and **Eli Shamovich**

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Example: $H^\infty(\mathfrak{B}_d)$. This operator-alg was extensively studied by

- Popescu as $\mathscr{F}_d^\infty$ and $H^\infty(B(\mathcal{X})_1^d)$;
- Davidson & Pitts as $\mathcal{L}_d$ – the nc analytic Toeplitz algebra;
- Muhly & Solel as $H^\infty(\mathbb{C}^d)$ – the Hardy algebra of $\mathbb{C}^d$.

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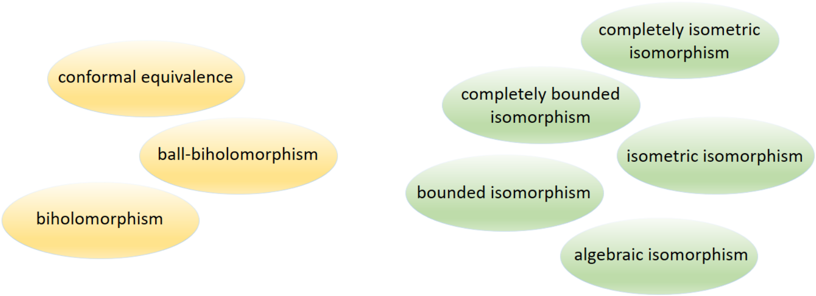

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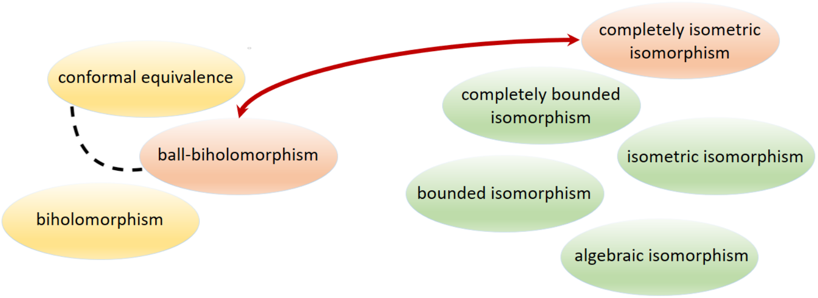

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Let $\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{B}_d$ *be nc varieties. There is a completely isometric isomorphism* $\alpha : H^\infty(\mathfrak{V}) \to H^\infty(\mathfrak{W})$ *if and only if* $\mathfrak{V}$ *and* $\mathfrak{W}$ *are ball-biholomorphic.*

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We conjectured that:

Ball-biholomorphism $\Longrightarrow$ **Conformal equivalence**

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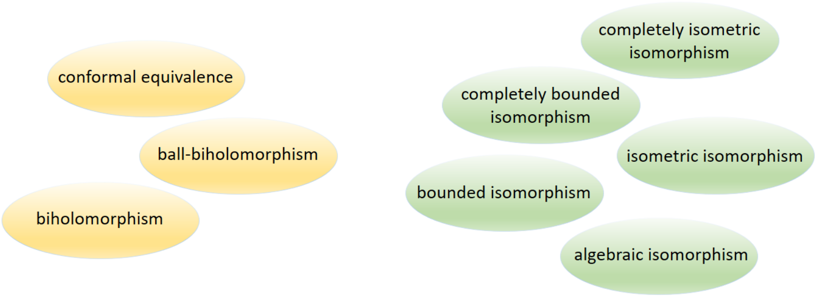

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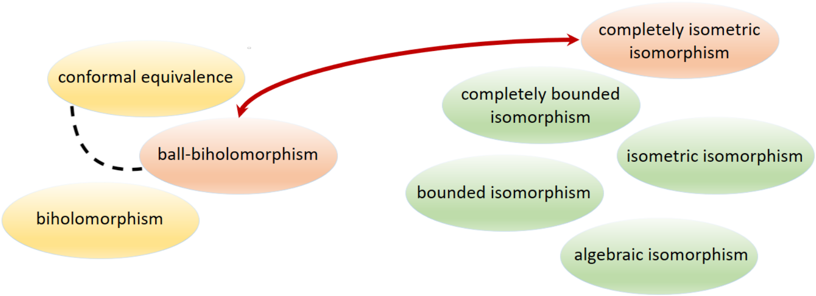

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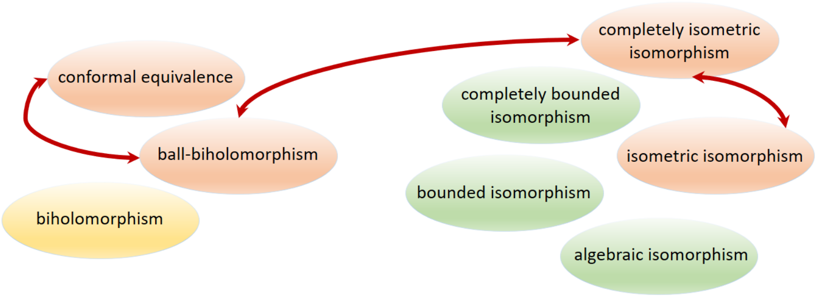

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Theorem (Salomon-S-Shamovich, 2017)

Let $\mathfrak{V} \subseteq \mathfrak{B}_d$ and $\mathfrak{W} \subseteq \mathfrak{B}_e$ be homogeneous nc varieties. Then TFAE:

- *$H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are completely isometrically isomorphic,*
- *$H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are isometrically isomorphic,*
- *$\mathfrak{V}$ and $\mathfrak{W}$ are ball-biholomorphic,*
- *$\mathfrak{V}$ and $\mathfrak{W}$ are conformally equivalent,*
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Proof: Basic nc function theory and tricks, analysis of fixed points of nc maps.

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Theorem (Shamovich, later in 2017)

Let $\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{B}_d$ be nc varieties which contain a scalar point.
If $\mathfrak{V}$ and $\mathfrak{W}$ are ball-biholomorphic $\Rightarrow$ conformally equivalent.

Shamovich's proof: Deeper analysis of fixed points of nc maps.

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These are not all algebraic automorphisms: if $g \in H^\infty(\mathfrak{B}_d)$ is invertible, then $f \mapsto gfg^{-1}$ is also an automorphism of $H^\infty(\mathfrak{B}_d)$.

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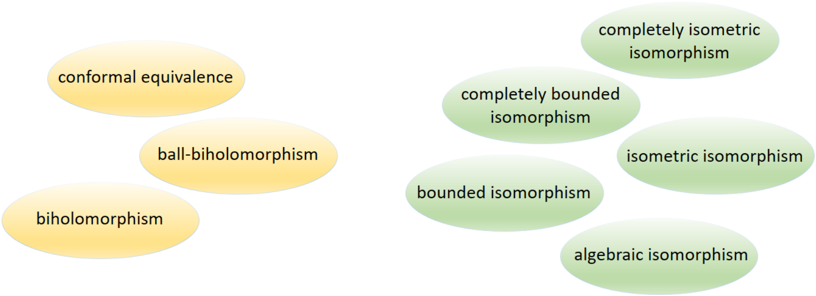

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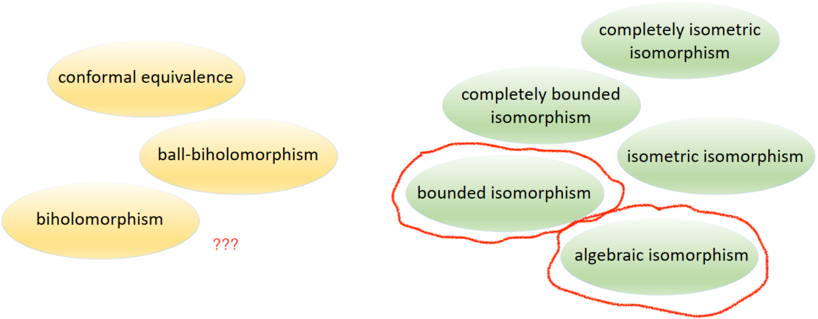

Similarity envelopes

The **similarity envelope** of a nc set Ω is

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G implements conjugation with g:

$$gfg^{-1} = f \circ G$$

Weak-$*$ continuous isomorphism

$$\widetilde{\mathfrak{V}} := \bigsqcup_{n=1}^{\infty} \left\{ S^{-1} X S : X \in \mathfrak{V}(n),\ S \in \mathrm{GL}_n(\mathbb{C}) \right\}.$$

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Theorem (Salomon-S-Shamovich)

Let $\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{B}_d$ be nc varieties. Then $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are weak-$*$ isomorphic if and only if $\widetilde{\mathfrak{V}}$ and $\widetilde{\mathfrak{W}}$ are biholomorphic via a nc map $G : \widetilde{\mathfrak{W}} \to \widetilde{\mathfrak{V}}$ satisfying

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Commutative nc setting (getting rid of weak-$*$)

$$\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{C}\mathbb{M}_d = \{X \in \mathbb{M}_d : X_i X_j = X_j X_i \text{ for all } i, j\}.$$

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Hope: this may shed light on the isomorphism problem in the fully commutative case.

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Theorem (Davidson & Pitts, 1998)

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- We do know that $\mathrm{Aut_b}(\widetilde{\mathfrak{B}}_d) \subsetneq \mathrm{Aut}(\widetilde{\mathfrak{B}}_d)$, e.g. take g unbounded.

Thank you