Dilations, inclusions of matrix convex sets, and completely positive maps

Ken Davidson, Adam Dor-On, Orr Shalit and Baruch Solel

Technion

July 2016
Dilations

\[ A = (A_1, \ldots, A_d) \in B(\mathcal{H})^d \]
\[ B = (B_1, \ldots, B_d) \in B(\mathcal{K})^d, \text{ where } \mathcal{K} \supset \mathcal{H} \]

**Definition**

A is said to be a **compression** of B if

\[ A_i = P_{\mathcal{H}} B_i |_{\mathcal{H}} \text{ for all } i = 1, \ldots d \]

We then say that B is a **dilation** of A, and we denote \( A \prec B \).
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Equivalently

\[ B_i = \begin{pmatrix} A_i & * \\ * & * \end{pmatrix} \]
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**Equivalently**

\[ \exists V : \mathcal{H} \rightarrow \mathcal{K} \text{ and isometry such that} \]

\[ A_i = V^* B_i V \]
Classical setting

\[ A = (A_1, \ldots, A_d) \in B(\mathcal{H})^d \text{ commuting operators} \]
\[ B = (B_1, \ldots, B_d) \in B(\mathcal{K})^d, \text{ normal commuting where } \mathcal{K} \supset \mathcal{H} \]

**Definition**

If for all \( n_1, \ldots, n_d \in \mathbb{N}, \)

\[ A_1^{n_1} \cdots A_d^{n_d} = P_\mathcal{H} B_1^{n_1} \cdots B_d^{n_d} |_{\mathcal{H}} \]

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$A = (A_1, \ldots, A_d) \in B(\mathcal{H})^d$ commuting operators
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$$B_1^{n_1} \cdots B_d^{n_d} = \begin{pmatrix} A_1^{n_1} \cdots A_d^{n_d} & * \\ * & * \end{pmatrix}$$

$$B_i = \begin{pmatrix} * & * & * \\ 0 & A_i & * \\ 0 & 0 & * \end{pmatrix}$$
Classical dilation theorems

**Theorem (Sz.-Nagy and Ando dilations theorem)**

For every pair of commuting contractions $A_1, A_2 \in B(\mathcal{H})$, there exists a pair of commuting unitaries $U_1, U_2 \in B(\mathcal{K})$ such that

$$A_1^m A_2^n = P_{\mathcal{H}} U_1^m U_2^n |_{\mathcal{H}} \quad \text{for all } m, n \geq 0$$
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For every $d$-tuple row contraction $(A_1, \ldots, A_d) \in B(\mathcal{H})^d$ (i.e., $\|[A_1 \cdots A_d]\| \leq 1$), there exists a $d$-tuple of isometries $V_1, \ldots, V_d \in B(\mathcal{K})$ with orthogonal ranges, such that for all $n_1, \ldots, n_k \in \{1, \ldots, d\}$,

$$A_{n_1} \cdots A_{n_k} = P_{\mathcal{H}} V_{n_1} \cdots V_{n_k} |_{\mathcal{H}}$$
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Theorem (Helton, Klep, McCullough, Schweighofer)

*Fix n and and a real n-dimensional Hilbert space $\mathcal{H}$.***
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Theorem (Helton, Klep, McCullough, Schweighofer)

Fix $n$ and and a real $n$-dimensional Hilbert space $\mathcal{H}$. There exists a constant $\vartheta_n$, a Hilbert space $\mathcal{K}$, and an isometry $V : \mathcal{H} \to \mathcal{K}$, and a commuting family $C$ in the unit ball of $B(\mathcal{K})_{sa}$ such that
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that for every contraction \( A \in B(\mathcal{H})_{sa} \), there exists \( N \in \mathcal{C} \) such that

\[
\frac{1}{\vartheta_n} A = V^* NV
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They also show find the optimal $\vartheta_n$, and show (!) $\vartheta_n \sim \frac{\sqrt{\pi n}}{2}$
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Questions:

• Why?
• Complex? (easy matter)
• Is there a constant independent of $n = \dim \mathcal{H}$? (must fix $d$)
• Can we obtain sharper control on the joint spectrum of $N$?
• If $d < \infty$ and $\dim \mathcal{H} < \infty$, can we do with $\dim \mathcal{K} < \infty$?
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Theorem (Davidson, Dor-On, S, Solel)

Suppose that \( K \subseteq \mathbb{R}^d \), with some nice symmetry properties. For every \( A \in B(\mathcal{H})_{sa}^d \) such that the numerical range \( \mathcal{W}_1(A) \subseteq K \), there is a \( d \)-tuple of commuting normal operators \( N \) on some Hilbert space \( \mathcal{K} \) such that

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\sigma(N) \subseteq K
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Normal dilation for noncommuting tuples

Rank independent dilation with spectral constraints

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If $\dim H < \infty$ then one can arrange $\dim K < \infty$
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If $\dim \mathcal{H} < \infty$ then one can arrange $\dim \mathcal{K} < \infty^a$

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$^a$General fact, here follows from explicit construction
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Explanations

Symmetry properties

There exist $k$ real $d \times d$ matrices $\lambda^{(1)}, \ldots, \lambda^{(k)}$ of rank one such that $I_d \in \text{conv}\{\lambda^{(1)}, \ldots, \lambda^{(k)}\}$ and

$$
\lambda^{(m)} K \subseteq dK, \quad m = 1, \ldots, k
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ETKIM: invariant under permutations and sign changes of coordinates. More generally: invariant under projections onto orthonormal basis.
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Numerical range

$$\mathcal{W}_1(A) := \{\phi(A) = (\phi(A_1), \ldots, \phi(A_d)) \mid \phi : B(\mathcal{H}) \to \mathbb{C} \text{ is a state} \}$$
Examples

Example - the regular tetrahedron

Not invariant under projection to an o.n.b.
Invariant under projections onto a frame.
(Also: convex hulls of frames with high symmetry, e.g., regular polytopes)

Image credit: http://mathworld.wolfram.com “Regular Tetrahedron”
Matrix sets

We are interested with working with all $d$-tuples of $n \times n$ matrices of all sizes:

$$\mathcal{M}^d := \bigcup_n (\mathcal{M}_n)^d$$

Another case of interest is all $d$-tuples of selfadjoint matrices of all sizes:

$$\mathcal{M}_{sa}^d := \bigcup_n (\mathcal{M}_n)_{sa}^d$$
Matrix sets

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Another case of interest is all $d$-tuples of selfadjoint matrices of all sizes:

$$\mathcal{M}_{sa}^d := \bigcup_n (M_n)_{sa}^d$$

A (free/matrix) subset of $\mathcal{M}_{sa}^d$ is a set of the form

$$S = \bigcup_n S_n$$

such that $S_n \subseteq (M_n)_{sa}^d$ for all $n$. 
Matrix convex sets

Definition

A set $S = \bigcup_n S_n \subseteq \mathcal{M}_s^d$ is said to be matrix convex if

1) $X \in S_n \subseteq (M_n)^d_s$ implies that for every $\phi \in \text{UCP}(M_n, M_m)$,

$$\phi(X) = (\phi(X_1), \ldots, \phi(X_d)) \in S$$

2) $X, Y \in S$ implies that

$$X \oplus Y = (X_1 \oplus Y_1, \ldots, X_d \oplus Y_d) \in S$$
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Equivalently, \( S \) is matrix convex if it is invariant under “matrix convex combinations”:

For \( X^{(i)} = (X_1^{(i)}, \ldots, X_d^{(i)}) \in S \) (for \( i = 1, 2, \ldots \))

\[
\sum_i (V_i^* X_1^{(i)} V_i, \ldots, V_i^* X_d^{(i)} V_i) \in S \quad \text{whenever} \quad \sum V_i^* V_i = I.
\]
Inclusions of matrix convex sets - motivation

Given two matrix convex sets $S = \bigcup_n S_n$ and $T = \bigcup_n T_n$, we say that $S \subseteq T$ if $S_n \subseteq T_n$ for all $n$.

Given two spectrahedra (convex sets) $K, L \subseteq \mathbb{R}^d$, determining whether $K \subseteq L$ is important in robust control, but can be computationally hard.

Let $S$ and $T$ be “free spectrahedra” (matrix convex sets) with $S_1 = K$ and $T_1 = L$.

[HKM2013] explain that considering inclusion problems of the type $S \subseteq T$ for the matrix convex sets, instead of inclusion problems $K \subseteq L$, generalizes a celebrated relaxation (which is a more tractable problem) of Ben-Tal and Nemirovski treating the special case $K = [-r, r]^d \subseteq L$.

Clearly $S \subseteq T \Rightarrow S_1 \subseteq T_1$.

Suppose we know that $S_1 \subseteq T_1$. What can we say?

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¹Does not seem to be of practical relevance to our work — more to [HKMS]
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Minimal and maximal matrix convex sets over $K$

Let $0 \in K \subseteq \mathbb{R}^d$ be a closed convex set. Then

$$K = \{ x \in \mathbb{R}^d : \sum a_i x_i \leq 1, \quad \forall (a_1, \ldots, a_d) \in K^\circ \}.$$
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We define

$$\mathcal{W}^{\max}(K) = \{ X \in \mathcal{M}^d_{sa} : \sum a_i X_i \leq I, \quad \forall (a_1, \ldots, a_d) \in K^\circ \}.$$ 

and

$$\mathcal{W}^{\min}(K) = \{ X \in \mathcal{M}^d_{sa} : \exists N \text{ normal } \sigma(N) \subseteq K \text{ and } X \prec T \}.$$
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$$\mathcal{W}^{\text{max}}(K) = \{ X \in \mathcal{M}^d_{sa} : \sum a_i X_i \leq I, \quad \forall (a_1, \ldots, a_d) \in K^\circ \}.$$ 

and

$$\mathcal{W}^{\text{min}}(K) = \{ X \in \mathcal{M}^d_{sa} : \exists N \text{ normal } \sigma(N) \subseteq K \text{ and } X \prec T \}.$$ 

$\mathcal{W}^{\text{min}}(K)$ and $\mathcal{W}^{\text{max}}(K)$ really are the minimal and maximal matrix convex sets “over” $K$. 

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Dilations, inclusions and CP maps
Application to matrix set inclusions

Theorem (DDSS, repeated)

Suppose that $K \subseteq \mathbb{R}^d$, with some nice symmetry properties. For every $A \in B(\mathcal{H})_{sa}^d$ such that the numerical range $\mathcal{W}_1(A) \subseteq K$, there is a $d$-tuple of commuting normal operators $N$, such that

$$\sigma(N) \subseteq K, \quad \frac{1}{d} A \prec N$$
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**Corollary**

$W^{max}(K) \subseteq dW^{min}(K)$. 
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$\mathcal{W}^{\text{max}}(K) \subseteq d\mathcal{W}^{\text{min}}(K)$. In particular, if $S_1 \subseteq T_1 = K$, then

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Sharpness: $d$ is the best constant that works for all such $K$. 

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Application to matrix set inclusions

**Theorem (DDSS, repeated)**

Suppose that \( K \subseteq \mathbb{R}^d \), with some nice symmetry properties. For every \( A \in B(\mathcal{H})_{sa}^d \) such that the numerical range \( \mathcal{W}_1(A) \subseteq K \), there is a \( d \)-tuple of commuting normal operators \( N \), such that

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\sigma(N) \subseteq K, \quad \frac{1}{d} A \prec N
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**Corollary**

\( \mathcal{W}^{\max}(K) \subseteq d \mathcal{W}^{\min}(K) \). In particular, if \( S_1 \subseteq T_1 = K \), then

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S \subseteq dT.
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**Sharpness:** \( d \) is the best constant that works for all such \( K \).

(we don’t know whether for all such \( K \), \( d \) is the best constant).
More on scaled dilations and scaled inclusions

Theorem (DDSS, following HKMS)

Let $K \subseteq \mathbb{R}^d$ and $c > 0$. TFAE:

1. For every $A \in \mathcal{M}_{sa}^d$ such that the numerical range $\mathcal{W}_1(A) \subseteq K$, there is a $d$-tuple of commuting normal operators $N$ on a finite dimensional Hilbert space, such that $\sigma(N) \subseteq K$ and $A \prec cN$.

2. $\mathcal{W}^{\text{max}}(K) \subseteq c\mathcal{W}^{\text{min}}(K)$.
More on scaled dilations and scaled inclusions

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Problem

Find the optimal constant $c$ for given $K \subseteq \mathbb{R}^d$. 

Sharper theorems, frames and polar duals — example

Denote $D = \{ x \in \mathbb{R}^d : \sum |x_i| \leq 1 \}$. By the corollary we have for $K = D$ or $K = [-1, 1]^d$, $\mathcal{W}^{\text{max}}(K) \subseteq d \mathcal{W}^{\text{min}}(K)$.
Sharper theorems, frames and polar duals — example

Denote $D = \{ x \in \mathbb{R}^d : \sum |x_i| \leq 1 \}$.

By the corollary we have for $K = D$ or $K = [-1, 1]^d$, 

$$\mathcal{W}^{\text{max}}(K) \subseteq d\mathcal{W}^{\text{min}}(K)$$

**Theorem (example)**

$$\mathcal{W}^{\text{max}}(D) \subseteq \mathcal{W}^{\text{min}}([-1, 1]^d), \quad \mathcal{W}^{\text{max}}([-1, 1]^d) \subseteq d\mathcal{W}^{\text{min}}(D).$$
Thank you!
Proof of the dilation theorem

Theorem (Davidson, Dor-On, S, Solel)

Suppose that $K \subseteq \mathbb{R}^d$, with some nice symmetry properties. For every $A \in B(\mathcal{H})_{sa}^d$ such that the numerical range $\mathcal{W}_1(A) \subseteq K$, there is a $d$-tuple of commuting normal operators $N$ on some Hilbert space $\mathcal{K}$ such that

$$\sigma(N) \subseteq K, \quad \frac{1}{d}A \prec N$$
Proof

Consider $A$ as a tuple of operators on a Hilbert space $\mathcal{H}$. Put $\mathcal{K} = \mathcal{H} \otimes \mathbb{C}^k$ and define $d^2$ diagonal, self-adjoint matrices $S_{i,j}$, $1 \leq i, j \leq d$, by

$$S_{i,j} = \text{diag}(\lambda_{i,j}^{(1)}, \ldots, \lambda_{i,j}^{(k)}).$$

(1)

Define

$$N_i = \sum_{j=1}^{d} A_j \otimes S_{i,j} \in B(\mathcal{K}), \ i = 1, \ldots, d.$$  

(2)
Proof

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$$S_{i,j} = \text{diag}(\lambda_{i,j}^{(1)}, \ldots, \lambda_{i,j}^{(k)}).$$  \hspace{1cm} (1)

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If $I_d = \sum_{p=1}^{k} \beta_p \lambda^{(p)}$, set $v = \sum_{p=1}^{k} \sqrt{\beta_p} e_p$ where $\{e_p\}$ is the standard basis of $\mathbb{C}^k$. 

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Dilations, inclusions and CP maps
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Consider $A$ as a tuple of operators on a Hilbert space $\mathcal{H}$. Put $\mathcal{K} = \mathcal{H} \otimes \mathbb{C}^k$ and define $d^2$ diagonal, self-adjoint matrices $S_{i,j}$, $1 \leq i, j \leq d$, by

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Define an isometry $V : \mathcal{H} \to \mathcal{K} = \mathcal{H} \otimes \mathbb{C}^k$ by $Vh = h \otimes v$. 

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The construction is complete. To finish the proof one checks that all claims hold.